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Percentage-Avoiding, Northwest Shapes and Peelable Tableaux

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We prove three results for Specht and Schur modules associated to *northwest shapes* and the more general class of %-*avoiding shapes*. The first result (conjectured for northwest shapes in by the authors) is a generalization the Littlewood–Richardson rule, giving an explicit combinatorial description for the multiplicities of irreducibles in the Specht and Schur modules of a %-avoiding shape D, in terms of D-*peelable tableaux*. The second result gives three involutions on the set of peelable tableaux which exhibit the symmetries of these multiplicities corresponding to three natural involutive operations on the set of %-avoiding shapes. The third result gives branching rules for the Specht and Schur modules of northwest shapes. The proofs are all combinatorial, with the exception of a key step in the first result, which requires results of Magyar on configuration varieties and characters of flagged Schur modules.

1. INTRODUCTION

To each *shape* D (finite subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$), one may associate the Specht module Sp_D, Schur module $\mathscr{S}_D$, and flagged Schur module $\mathscr{S}_D^{\mathrm{flag}}$, which carry actions of the symmetric group, general linear group, and Borel subgroup of lower triangular matrices, respectively. Specht and Schur modules

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were originally introduced by Young and Weyl for Ferrers shapes of partitions, as a means for explicitly producing the irreducible representations of the symmetric and general linear groups. They are also well-studied for *skew Ferrers shapes*, where their decomposition into irreducibles is given by the well-known *Littlewood–Richardson rule* (see, e.g., [8, 12, 27, Section 4.9]). More general classes of shapes were studied in [1, 8, 10, 11, 19, 20–22, 24, 25], where they arise naturally in connection with resolutions of determinantal ideals and in the theory of Schubert polynomials and reduced decompositions of permutations. The characters of flagged Schur modules include key polynomials (type A Demazure characters), flagged skew Schur functions, and Schubert polynomials.

This paper studies the Specht, Schur, and flagged Schur modules associated to a large family of shapes we call %-*avoiding*. Informally, a shape is %-avoiding if it does not have a pair of rows and a pair of columns such that its restriction to these rows and columns looks like

$$\begin{matrix} \cdot & \times \\ \times & \cdot \end{matrix},$$

where the symbol $\times$ indicates a cell that is present in a shape and $\cdot$ indicates a cell that is absent. (We call them %-avoiding due to the vague resemblance of this forbidden diagram with the percent symbol). This family of shapes includes almost all others previously considered, including Ferrers and skew Ferrers shapes, column-convex shapes, Rothe diagrams of permutations, northwest shapes [24, 25], and the complements of all these kinds of shapes in an enclosing rectangle [19]. The only other family for which some results are known are the three-rowed shapes [22].

We introduce the notions of peelable tableaux for %-avoiding shapes and corner cells and horizontal strips for northwest shapes, in order to prove these main results.

THEOREM 1. *Let D be a %-avoiding shape. Then the Specht module* Sp_D *and Schur module* $\mathcal{S}_D$ *have the following decomposition into irreducible representations*:

$$\mathrm{Sp}_D \cong \bigoplus_{D\text{-peelable tableaux } Q} \mathrm{Sp}_{\mathrm{shape}(Q)}$$

$$\mathcal{S}_D \cong \bigoplus_{D\text{-peelable tableaux } Q} \mathcal{S}_{\mathrm{shape}(Q)}.$$

Theorem 1 is proven using a new combinatorial decomposition of the character of a flagged Schur module of a %-avoiding shape (see Theorem 20).

THEOREM 2. *Let i be any of the following three involutions on the family of %-avoiding shapes*:

- Evac: *rotation through* 180°.
- Tr: *transposition across the diagonal.*
- Boxcomp: *set complementation within an enclosing rectangular shape followed by left-to-right reflection.*

Then there is a corresponding involution i that maps the D-peelable tableaux bijectively to the i(D)-peelable tableaux. Furthermore, these involutions on tableaux satisfy the same commutation relations as they do on shapes, namely,

$$\text{Evac Tr} = \text{Tr Evac}$$

$$\text{Evac Boxcomp} = \text{Boxcomp Evac}$$

$$\text{Tr Boxcomp} = \text{Evac Boxcomp Tr.}$$

These involutions generate a dihedral group of order 8.

THEOREM 3. *Let D be a northwest shape of cardinality n with at most N rows. Then there is an isomorphism of $\sum_{n-1}$-modules,*

$$\operatorname{Res}^{\Sigma_n}_{\Sigma_{n-1}} \operatorname{Sp}_D \cong \bigoplus_{x \text{ a corner cell of } D} \operatorname{Sp}_{D-x},$$

and an isomorphism of GL_{N-1}-modules,

$$\operatorname{Res}^{GL_N}_{GL_{N-1}} \mathcal{S}_D \cong \bigoplus_{X \text{ a horizontal strip of } D} \mathcal{S}_{D-X}.$$

There are two previously known families of shapes D for which D-peelability has a simpler description. For *column-convex* shapes D, D-peelability is equivalent to *D-decomposability* [25]. Specializing further, for skew shapes D, the D-decomposable tableaux biject with the tableaux that appear in the usual Littlewood–Richardson rule [12]. For the Rothe diagram $D(w)$ of a permutation w [18, p. 8], it is shown in [24] that D-peelable tableaux are in shape-preserving bijection with the reduced word column-strict tableaux of Edelman and Greene [4] and Lascoux and Schützenberger [17].

The paper is structured as follows. Section 2 recalls definitions of various notions about tableaux, and the constructions for Specht, Schur, and flagged Schur modules. Sections 3, 4, and 5 contain full statements of the three main results with definitions and examples of D-peelable tableaux, the involutions Evac, Tr, Boxcomp, corner cells, and horizontal and vertical strips. Section 6 is devoted to remarks and open problems. Appendicies A1 through A4 contain all proofs.

2. DEFINITIONS

A *shape* D is a finite subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$; its elements are called *cells*. We adopt the English notation for depicting shapes. The cell (i, j) is viewed as the position in a matrix in the ith row from the top and the jth column from the left. The jth column of a shape D will often be viewed as the set $\{i: (i, j) \in D\}$ of the row indices of its cells. A similar identification will be made for a row of a shape. Unless explicitly stated otherwise, we only consider shapes that are contained in a fixed $r \times c$ rectangular shape. The product of symmetric groups $\sum_r \times \sum_c$ acts on shapes D by permuting the row and column indices of cells: $(u, v)\, D = \{(u(i), v(j)) : (i, j) \in D\}$.

A *composition* $\alpha = (\alpha_1, \alpha_2, ...)$ is a sequence of nonnegative integers, almost all zero. Let $D(\alpha)$ be the shape whose ith row consists of α_i left-justified cells for all i. A *partition* $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l)$ is a weakly decreasing composition. We often identify a partition λ with its *Ferrers shape* $D(\lambda)$. A *skew shape* λ/μ is the set difference $D(\lambda) - D(\mu)$. A *horizontal strip* (resp. *vertical strip*) is a skew shape such every column (resp. row) contains at most one cell.

An *alphabet* is a totally ordered set; its elements are called *letters*. The default alphabet is $\mathbb{Z}_+$. A *word* is a finite sequence of letters. A subword is a (not necessarily contiguous) subsequence. If a is a word in the alphabet A and $B \subset A$ is a subalphabet, let $a|_B$ denote the word obtained from a by erasing its letters that are not in B. If A and B are alphabets, $A + B$ is the alphabet with set $A \cup B$ such that $a < b$ for all $a \in A$ and $b \in B$.

Important convention. If C is a finite alphabet, we often write C to denote the word comprised of the letters of C listed in *decreasing* order.

The exponential notation a^m stands for the word comprised of m copies of the letter a. The interval of integers weakly between m and n is denoted $[m, n]$, and $[1, n]$ is abbreviated to $[n]$. The *content* of a word b in the alphabet $\mathbb{Z}_+$, denoted content(b), is the composition whose ith part is the number of occurrences of the letter i in b.

A *filling* T of D in the alphabet A is an assignment $T: D \to A$ which can be viewed as a partial matrix whose (i, j)th position contains the value $T(i, j) \in A$ for all $(i, j) \in D$. Let $\mathrm{Row}_D: D \to \mathbb{Z}_+$ be the filling of the shape D such that every cell in the ith row of D contains the value i.

The Specht, Schur, and flagged Schur modules have the following well-known constructions. For ease of exposition, it will be assumed that the base ring is a field of characteristic zero. see Section 6 for comments on other base rings.

Consider the polynomial ring $\mathbb{F}[z_{ij}]_{i,\, j \geqslant 1}$. The matrix $g = (g_{ij}) \in GL(N)$ acts on $\mathbb{F}[z_{ij}]$ by

$$gz_{kl} = \begin{cases} \sum_{i=1}^{N} g_{ik} z_{il} & \text{for} \quad k \leqslant N \\ z_{kl} & \text{otherwise.} \end{cases}$$

The symmetric group $\sum_n$ acts on $\mathbb{F}[z_{ij}]$ by the subgroup of permutation matrices in $GL(n)$.

For a pair of words $u = u_1 u_2 \cdots u_p$ and $v = v_1 v_2 \cdots v_p$ of equal length, let

$$(u \mid v) = \det(z_{u_i,\, v_j})_{1 \leqslant i,\, j \leqslant p}$$

denote the minor of the generic matrix (z_{ij}) whose row and column indices are given by u and v, respectively. For a filling T of D, let

$$\Delta_T = \prod_j \, (u^{(j)} \mid v^{(j)}), \tag{2.2.1}$$

where $u^{(j)}$ and $v^{(j)}$ are the jth columns of T and Row_D, respectively. Clearly $\Delta_T = 0$ if some column of T has a repeat, and $\Delta_T = \pm \Delta_{T'}$, where T' is obtained by sorting the columns of T into increasing order from top to bottom.

EXAMPLE 4. Here is a filling T of D, the filling Row_D, and the element Δ_T:

$$T = \begin{matrix} 1 & \cdot & \cdot \\ 3 & \cdot & 3 \\ \cdot & 2 & 4 \end{matrix} \qquad \mathrm{Row}_D = \begin{matrix} 1 & \cdot & \cdot \\ 2 & \cdot & 2 \\ \cdot & 3 & 3 \end{matrix} \qquad \Delta_T = \begin{vmatrix} z_{11} & z_{12} \\ z_{31} & z_{32} \end{vmatrix} \, |z_{23}| \, \begin{vmatrix} z_{32} & z_{33} \\ z_{42} & z_{43} \end{vmatrix}.$$

DEFINITION 5. The Schur module $\mathcal{S}_D$ is the $\mathbb{F}$-linear span of the elements Δ_T, where T runs over the fillings of D whose values do not exceed N.

By the multilinearity of the determinant it is easy to see that $\mathcal{S}_D$ is stable under the action of $GL(N)$.

DEFINITION 6. Let n be the cardinality of D. The Specht module Sp_D is the $\mathbb{F}$-linear span of the elements Δ_T, where $T\colon D \to [n]$ is a bijection.

The symmetric group $\sum_n$ stabilizes Sp_D due to the formula $\sigma\Delta_T = \Delta_{\sigma \circ T}$.

Let B be the subgroup of lower triangular matrices in $GL(N)$. Consider the B-stable ideal I in $\mathbb{F}[z_{ij}]$ generated by z_{ij} for $i > j$. It is not hard to show that for strictly increasing words u and v, $(u \mid v) \in I$ if and only if $u_i > v_i$ for some i.

Definition 7. The flagged Schur module $\mathscr{S}_D^{\mathrm{flag}}$ is the B-module $\mathscr{S}_D/(\mathscr{S}_D \cap I)$.

Say that a filling T of D is *flagged* if its columns are strictly increasing, and for all i the value of each letter in its ith row does not exceed i. If T is a filling of D that is not flagged, then it is easy to see that $\Delta_T \in I$. Thus $\mathscr{S}_D^{\mathrm{flag}}$ is spanned by the elements Δ_T where T runs over the flagged fillings of D.

It is easy to see that Sd_D and $\mathscr{S}_D$ are unchanged up to isomorphism by row and column rearrangements. $\mathscr{S}_D^{\mathrm{flag}}$ is unchanged up to isomorphism by column rearrangements. Since $\mathbb{F}$ is a field of characteristic zero, the Specht module Sp_D may be decomposed into irreducible representations of $\sum_n$, which are isomorphic to Sp_λ for various partitions λ of n (see, e.g., [27, Section 2.4]). Define the multiplicities c_λ^D by

$$\mathrm{Sp}_D \cong \bigoplus_\lambda (\mathrm{Sp}_\lambda)^{\oplus c_\lambda^D}. \tag{2.2.2}$$

Similarly, the Schur module $\mathscr{S}_D$ is a polynomial representation of $GL(N, \mathbb{F})$ of degree n and, hence, may be decomposed into irreducible polynomial representations of $GL(N, \mathbb{F})$ of degree n, which are isomorphic to $\mathscr{S}_\lambda$ for various partitions λ of n of length at most N (see, e.g., [5, Part I, Section 6]),

$$\mathscr{S}_D \cong \bigoplus_\lambda (\mathscr{S}_\lambda)^{\oplus c_\lambda^D} \tag{2.2.3}$$

for some nonnegative integers c_λ^D. It is a consequence of *Schur–Weyl duality* (see [5]) that the integers c_λ^D that occur in Eqs. (2.2.2) and (2.2.3) are the same, provided the number of nonempty rows of D does not exceed N. It is a central problem in the study of Specht and Schur modules to interpret these coefficients c_λ^D for various shapes D. The most well-known example is the Littlewood–Richardson rule for skew shapes [27, Section 4.9]. Other results interpreting c_λ^D for various classes of shapes D were given in [24, 25]. All of these results are generalized by Theorem 1.

The computation of the multiplicities c_λ^D in (2.2.3) is facilitated by means of formal characters. The formal *character* of a finite dimensional B-module M is defined to be the polynomial in the variables $x_1, x_2, \ldots, x_N$ given by the trace of the action of the diagonal matrix $x = \operatorname{diag}(x_1, ..., x_N)$ on M. It follows from (2.2.3) that $\operatorname{char}(\mathcal{S}_D)$ has a unique expansion

$$\operatorname{char}(\mathcal{S}_D)(x_1, ..., x_N) = \sum_\lambda c_\lambda^D s_\lambda(x_1, ..., x_N), \tag{2.2.4}$$

where s_λ is the *Schur polynomial* corresponding to λ [27, Section 4.4].

We next review some facts about tableaux, plactic equivalence, Schensted column insertion, and Schützenberger's jeu de taquin. A good reference for much of this material is [27].

Following [9], define the *Knuth* or *plactic* equivalence relation $\sim$ on words in an alphabet to be the transitive closure of the *Knuth* transformations

$$\begin{aligned} &uikjv \sim ukijv \qquad \text{for letters} \quad i \leqslant j < k \\ &ujikv \sim ujkiv \qquad \text{for letters} \quad i < j \leqslant k, \end{aligned}$$

where u and v are arbitrary words. Note that if $a \sim b$ then $a|_B \sim b|_B$, where B is a subinterval of the alphabet of the words a and b.

A *column-strict tableau* T of skew shape λ/μ is a filling of λ/μ that is weakly increasing from left to right in each row, and strictly increasing from top to bottom in each column. We will often refer to column-strict tableaux as simply *tableaux*. The *row-reading word* $\operatorname{rowword}(T)$ of a tableau T is the word obtained by reading the entries from left to right in each row, starting with the bottom row and proceeding toward the top. The column-reading word $\operatorname{columnword}(T)$ of a tableau T is that obtained by reading the entries from bottom to top in each column, starting with the leftmost column and proceeding rightward. It is well known that the row- and column-reading word of a tableau are Knuth equivalent. Often when we are only concerned with the Knuth equivalence class of a tableau T, we will often write T in place of its row- or column-reading word. For the word b, let $P(b)$ be the unique tableau of partition shape whose row- or column-reading word is Knuth equivalent to b.

A *standard word* of length n is one that contains each of the integers $1, 2, ..., n$ exactly once. A *descent* (resp. *ascent*) of a standard word a is a number k such that the $(k+1)$th letter in a is smaller (resp. larger) than the kth. A *retreat* (resp. *advance*) of the standard word a is a number k such that $k+1$ appears to the left (resp. right) of k in a. A *standard tableau* is a column-strict tableau whose word is standard. A *descent* (resp. *ascent*) of a standard tableau S is a number k such that $k+1$ appears in a later (resp. weakly earlier) row in S than k does. Note that k is a descent of the standard tableau S if and only if k is a retreat of $\operatorname{rowword}(S)$.

There is an algorithm called *Schensted column insertion* of the letter x into the tableau (of partition shape) T that computes the tableau $P(xT)$.

Let $a_k a_{k-1} \cdots a_1$ be the strictly decreasing word given by the first column of T:

1. If $x > a_k$, then $P(xT)$ is the tableau obtained by adjoining the letter x to the bottom of the first column of T.

2. Otherwise, let i be minimal such that $x \leqslant a_i$. In this case it is said that the letter x "bumps" the letter a_i. Let $\hat{T}$ be the tableau obtained by removing the first column from T. Then $P(xT)$ is the tableau whose first column is given by the word $a_k \cdots a_{i+1} x a_{i-1} \cdots a_2 a_1$ and whose remainder is given by $P(a_i \hat{T})$.

The *bumping path* of the column insertion of x into T either refers to the (weakly increasing) sequence of letters (starting with x) that are bumped, or to the sequence of cells containing the bumped letters. This sequence of cells always proceeds strictly east and weakly north.

This algorithm leads to the following bijection from words b to pairs of tableaux $(P(b), Q(b))$ of the same shape, where $P = P(b)$ is column-strict and $Q = Q(b)$ is standard. Let $b = b_n b_{n-1} \cdots b_1$ be a word. Define the tableaux P_i by $P_0 = \varnothing$ (the empty tableau) and $P_i = P(b_i P_{i-1})$ for $1 \leqslant i \leqslant n$. Let $P = P_n$ (so that $P = P(b)$ as defined above) and let Q be the standard tableau of the same shape as P such that the cell shape(P_i)/shape(P_{i-1}) in Q contains the letter i.

The column insertion algorithm can be "undone"; the inverse algorithm is called reverse column insertion. Given a tableau S of partition shape λ and a corner cell s of λ (a cell of the form $s = (i, \lambda_i)$, where $\lambda_i > \lambda_{i+1}$), there is a unique tableau T of the partition shape $\lambda - s$ and a unique value x such that such that $P(xT) = S$. The tableau T is referred to as the result of the reverse column insertion on S at s, and x is called the ejected letter.

Pieri's rule for column insertion states that if $b = b_n \cdots b_1$, then i is a descent of $Q(b)$ if and only if $b_{i+1} > b_i$, and that in this case the bumping paths of b_i and b_{i+1} are disjoint.

There is another algorithm known as *Schensted rove insertion* of the letter x into the tableau T of partition shape, which calculates the tableau $P(Tx)$. It leads to another bijection from words $c = c_1 c_2 \cdots c_n$, to pairs of tableaux (P, Q), where P is column-strict and Q standard, both of the same partition shape, given by $P = P(c)$ and Q contains the letter i in the cell shape$(P(c_1 \cdots c_{i-1}))$/shape$(P(c_1 \cdots c_i))$. Row insertion can also be reversed.

Schensted defined a *standardization* map from words to standard words and (skew) tableaux to (skew) standard tableaux. Given a word $a = a_1 a_2 \cdots a_n$, let $\sigma = \mathrm{std}(a)$ be the unique standard word of length n such that $\sigma_i < \sigma_j$ if $a_i < a_j$ or if $a_i = a_j$ and $i < j$. Clearly σ is the standardization of a word of content α if and only if every retreat of σ has the form $\alpha_1 + \alpha_2 + \cdots + \alpha_i$ for some i. In this situation, let $\mathrm{std}^{-1}(\sigma, \alpha)$ be the unique word of content α

whose standardization is σ. It follows from the definitions that $a \sim b$ if and only if $\mathrm{std}(a) \sim \mathrm{std}(b)$ and $\mathrm{content}(a) = \mathrm{content}(b)$. If T is a (skew) tableau, define $\mathrm{std}(T)$ to be the unique tableau of the same shape as T such that $\mathrm{rowword}(\mathrm{std}(T)) = \mathrm{std}(\mathrm{rowword}(T))$ (the column-reading word could also be used, with the same result). It can be shown that $\mathrm{std}(T)$ is a standard tableau, $P(\mathrm{std}(a)) = \mathrm{std}(P(a))$, and $Q(\mathrm{std}(a)) = Q(a)$. For a (skew) standard tableaux S, let $\mathrm{std}^{-1}(S, \alpha)$ be defined in the obvious way.

There is a related algorithm called Schützenberger's *jeu de taquin* [27, Section 3.9], which sends a (skew) tableau to a Knuth equivalent (skew) tableau of a different shape. Say that the skew shape E *extends* the skew shape D if there are partitions $v \subseteq \mu \subseteq \lambda$ such that $D = \mu/v$ and $E = \lambda/\mu$. In this case let $D + E$ denote the skew shape λ/v. If E extends D, and if S and T are tableaux of shapes D and E in the alphabets A and B, respectively, let $S + T$ denote the tableau of shape $D + E$ in the alphabet $A + B$ whose restrictions to the subshapes D and E are given by S and T, respectively. Suppose that S and T are standard tableaux such that shape(S) extends shape(T). Let $j_S(T)$ (resp. $j^T(S)$) denote the standard tableau obtained from sliding T (resp. S) to the southeast (resp. northwest) into the sequence (resp. reverse sequence) of cells given by S (resp. T). Write $v_S(T)$ (resp. $v^T(S)$) for the standard tableau given by the sequence (resp. reverse sequence) of cells vacated by the operation $j_S(T)$ (resp. $j^S(T)$). One has the following facts:

1. $j_S(T)$ and T (resp. $j^T(S)$ and S) are Knuth equivalent, and therefore have the same descents (just restrict to two-letter subintervals).
2. $v_S(T) = j^T(S)$ and $v^T(S) = j_S(T)$.

Now let S and T be column-strict tableaux such that shape(S) extends shape(T). Let

$$j_S(T) = \mathrm{std}^{-1}(j_{\mathrm{std}(S)}(\mathrm{std}(T)), \mathrm{content}(T))$$

$$v_S(T) = \mathrm{std}^{-1}(v_{\mathrm{std}(S)}(\mathrm{std}(T)), \mathrm{content}(T))$$

$$j^T(S) = \mathrm{std}^{-1}(j^{\mathrm{std}(T)}(\mathrm{std}(S)), \mathrm{content}(S))$$

$$v^T(S) = \mathrm{std}^{-1}(v^{\mathrm{std}(T)}(\mathrm{std}(S)), \mathrm{content}(S)).$$

Then 1 and 2 also hold when S and T are column-strict.

Necessary for the definition of peelability is Lascoux and Schützenberger's plactic action of the symmetric group on words [14]. Given *a* word b, the *i-pairing* of b is the well-formed parenthesization that comes from labelling the i's (resp. $i + 1$'s) in b with right parentheses (resp. left parentheses), and only considering the parentheses that close each other. For example, the 2-pairing of a word is computed below.

$$\begin{array}{ccccccccccccccccccccc}
5 & 2 & 4 & 2 & 3 & 4 & 3 & 2 & 4 & 5 & 1 & 2 & 4 & 2 & 3 & 5 & 5 & 2 & 6 & 4 & 3 \\
 &) & &) & (& & (&) & & & &) & &) & (& & &) & & & (\\
 & & & & (& & (&) & & & &) & & & (& & &) & & &
\end{array}$$

Note that the *i-unpaired* i's and $i+1$'s in b (i.e., those that are not in the i-pairing of b) must appear in b as a subsequence of the form $i^k(i+1)^l$. Given a word b with i-unpaired subword $i^k(i+1)^l$, let $e_i(b)$ and $f_i(b)$ be the words obtained from b by replacing the i-unpaired subword by $i^{k+1}(i+1)^{l-1}$ and $i^{k-1}(i+1)^{l+1}$, respectively, leaving all other letters unchanged. Note that $e_i(b)$ (resp. $f_i(b)$) is defined only if $l>0$ (resp. $k>0$). Define the *plastic transposition* s_i [14, Section 4.3] by $s_i(b)=e_i^{l-k}(b)$ if $k\leqslant l$ and $s_i(b)=f_i^{k-l}(b)$ if $k\geqslant l$. The operations e_i, f_i, and s_i do not change the positions of i-unpaired letters in a word.

Continuing our previous example, we have $k=3$, $l=1$, and

$$b=5\mathbf{2}4\mathbf{2}\mathbf{3}4\mathbf{3}\mathbf{2}451\mathbf{2}4\mathbf{2}\mathbf{3}55\mathbf{2}64\mathbf{3}$$

$$s_2(b)=5\mathbf{2}4\mathbf{3}\mathbf{3}4\mathbf{3}\mathbf{2}451\mathbf{2}4\mathbf{3}\mathbf{3}55\mathbf{2}64\mathbf{3}.$$

It is not hard to show that if F is any of the operators e_i, f_i, or s_i, then

Plac1. If $b=\operatorname{rowword}(T)$, where T is a (skew) tableau and $F(b)$ is defined, then there is a unique tableau (call it $F(T)$) such that $\operatorname{shape}(F(T))=\operatorname{shape}(T)$ and $\operatorname{rowword}(F(T))=F(\operatorname{rowword}(T))$.

Plac2. If $a\sim b$ then $F(a)$ is defined if and only if $F(b)$ is, and $F(a)\sim F(b)$. Furthermore, $Q(F(a))=Q(a)$.

Plac3. The operators $\{s_i\}$ satisfy the Moore–Coxeter relations

$$s_i^2=\text{identity}$$

$$s_is_j=s_js_i \qquad \text{if} \quad |i-j|>1$$

$$s_is_{i+1}s_i=s_{i+1}s_is_{i+1}.$$

Thus one may define the *plactic action* $u(b)$ of the permutation u on the word b by $u(b)=s_{i_1}\cdots s_{i_k}b$, where $u=s_{i_1}\cdots s_{i_k}$ is any shortest expression for u as a product of adjacent tranpositions $s_i=(i\ \ i+1)$.

We will usually omit the parentheses in the notation for the action of a plactic permutation. In the absense of parentheses, the plastic permutation acts on the entire subword to its right.

Lastly, we review the notions of key polynomials, key tableaux, the left and right key of a tableau, and compatible sequences [23]. Let $\alpha=(\alpha_1, \alpha_2, ...)$

be a composition. The *key polynomial* κ_α of Lascoux and Schützenberger [16], or the *Demazure character* [2], is defined by the recurrence

$$\kappa_\alpha = \begin{cases} x^\alpha & \text{if} \quad \alpha_1 \geqslant \alpha_2 \geqslant \cdots \\ \pi_i \kappa_{s_i\alpha} & \text{if} \quad \alpha_i < \alpha_{i+1}, \end{cases} \tag{2.2.5}$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, the composition $s_i\alpha$ is obtained from α by exchanging α_i and α_{i+1}, and π_i is the *isobaric divided difference* or *Demazure operator*,

$$\pi_i(f) = \frac{\pi_i f(x) - x_{i+1} s_i f(x)}{x_i - x_{i+1}}, \tag{2.2.6}$$

where the adjacent transposition s_i acts on the polynomial f by substituting x_i by x_{i+1} and vice versa. It can be shown that the definition (2.2.5) does not depend on the choice of i.

The first combinatorial description of κ_α involves key tableaux. A *key tableau* is a column-strict tableau of partition shape such that the set of elements C_j in the jth column of K satisfy the inclusions $\cdots \subseteq C_3 \subseteq C_2 \subseteq C_1$. There is a bijection $K \mapsto \alpha$ from the set of key tableaux to the set of compositions, sending a tableau to its content. Denote by key(α) the unique key tableau of content α. Let Q be a column-strict tableau of the partition shape λ. The *left key* $K_- Q$ (resp. *right key* $K_+ Q$) of Q is defined to be the key tableau of shape λ, whose jth column is comprised of the letters that are ejected during successive reverse column (resp. row) insertions on Q at the sequence of cells $(c, \lambda_c), (c-1, \lambda_{c-1}), ..., (1, \lambda_1)$, where c is the length of the jth column of λ. See [16; 23, p. 112; 24, p. 345] for other methods for calculating the left and right keys. The key polynomial is given by [16]

$$\kappa_\alpha = \sum_{K_+ P \leqslant \operatorname{key}(\alpha)} x^{\operatorname{content}(P)}, \tag{2.2.7}$$

where $\leqslant$ means entrywise comparison of key tableaux.

Another description of κ_α is given by *compatible sequences*. Given a word $a = a_1 a_2 \cdots a_n$, an *a-compatible sequence* is a word $i_1 i_2 \cdots i_n$ such that

C1. $i_j \leqslant a_j$ for all j.

C2. $i_j \leqslant i_{j+1}$ for all j.

C3. $i_j < i_{j+1}$ whenever $a_j < a_{j+1}$.

It is proven in [23, Theorem 5] that

$$\kappa_\alpha = \sum_{\substack{\operatorname{rev}(a) \sim \operatorname{key}(\alpha) \\ i \text{ is a-compatible}}} x^{\operatorname{content}(i)},$$

where $\mathrm{rev}(a)$ is the *reverse* $a_n a_{n-1} \cdots a_2 a_1$ of the word a. More generally it is shown there that for any column-strict tableau Q of partition shape,

$$\kappa_{\mathrm{content}(K\_Q)} = \sum_{\substack{\mathrm{rev}(a) \sim Q \\ i \text{ is a } a\text{-compatible}}} x^{\mathrm{content}(i)}. \tag{2.2.8}$$

Finally, κ_α is the character of the flagged Schur module $\mathscr{S}^{\mathrm{flag}}_{D(\alpha)}$, also known as a *Demazure module*.

THEOREM 8 (see [2]). $\mathrm{char}(\mathscr{S}^{\mathrm{flag}}_{D(\alpha)}) = \kappa_\alpha$

3. DECOMPOSITION INTO IRREDUCIBLES FOR %-AVOIDING SHAPES

The goal of this section is to define %-avoiding shapes and peelable tableaux, to restate Theorem 1, and to illustrate some of these notions with examples. Proofs are deferred to Appendices A1 and A3.

DEFINITION 9. Say that a shape D is *%-avoiding* if whenever $(j, k), (i, l) \in D$ with $i < j$ and $k < l$, then either $(i, k) \in D$ or $(j, l) \in D$. Say that the shape D is *northwest* if whenever $(j, k), (i, l) \in D$ with $i < j$ and $k < l$, then $(i, k) \in D$.

The family of %-avoiding shapes contains all northwest shapes, and in particular contains the Ferrers shapes and the left-to-right reflections of skew shapes.

EXAMPLE 10. Let D_1, D_2, D_3, D_4, D_5 be the following shapes:

$$\begin{array}{cccc} \times & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array}, \quad \begin{array}{cccc} \times & \times & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot \\ \cdot & \times & \times & \times \\ \cdot & \times & \cdot & \cdot \\ \cdot & \cdot & \times & \cdot \end{array}, \quad \begin{array}{cc} \cdot & \times \\ \times & \cdot \end{array}, \quad \begin{array}{cc} \times & \cdot \\ \cdot & \times \\ \times & \cdot \end{array}, \quad \begin{array}{cccccc} \times & \cdot & \cdot & \times & \times & \cdot \\ \cdot & \times & \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \cdot & \times & \times \end{array}$$

D_1 is %-avoiding, but not northwest. D_2 is northwest. D_3 is not %-avoiding, but after exchanging its columns it becomes northwest, so that all of our results are still applicable. D_4 is not %-avoiding, but after exchanging its last two rows it becomes northwest, so that all of our results, except for those involving $\mathscr{S}^{\mathrm{flag}}_{D}$ are applicable. D_5 is not %-avoiding, and it is impossible to permute its rows and columns in any way to make it so. This is the smallest shape with this property.

Consider the *orthodontic* partial order on the set of all shapes, whose covering relation is defined by the following cases:

Or1. If the leftmost nonempty column C of D is an initial segment, then

$$D - C <_{ortho} D.$$

Or2. If the ith row of D is properly contained in the $(i+1)$th, then

$$(s_i, id)\, D <_{ortho} D.$$

Remark 11. It follows directly from the definitions that if $D' <_{ortho} D$, then D is %-avoiding if and only if D' is.

PROPOSITION 12. *D is %-avoiding if and only if $\varnothing \leqslant_{ortho} D$, where $\varnothing$ is the empty shape.*

We will refer to the restriction of the orthodontic partial order to the family of %-avoiding shapes, as the *orthodontic poset*.

DEFINITION-PROPOSITION 13. Let D be a %-avoiding shape and Q a column-strict tableau of partition shape. There is a well-defined notion of a D-peelable tableau Q given by induction on the orthodontic partial order as follows:

P1. The empty tableau is the unique peelable tableau for the empty shape.

P2. If the first nonempty column C of D is an initial segment (that is, $C = [k]$ for some $k \geqslant 1$) then Q is D-peelable if and only if the first column of Q contains C and $P(Q - C)$ is $(D - C)$-peelable.

P3. If the ith row of D is properly contained in the $(i+1)$th, then Q is D-peelable if and only if $s_i Q$ is $(s_i, id)\, D$-peelable.

Say that the word a is D-peelable if and only if $P(a)$ is D-peelable.

EXAMPLE 14. For the shape D_1 of Example 10, the D_1-peelable tableaux are

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & 3 & \end{matrix}\ ,\quad \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & & \end{matrix},\quad \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{matrix}\ ,\quad \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 3 & & \\ 3 & & & \end{matrix}\ ,\quad \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \end{matrix} \right\}.$$

Let Q_1 be the third tableau on this list. Its D_1-peelability is exhibited by a saturated chain of shapes in the orthodontic poset going from D_1 down to $\varnothing$. Each shape is accompanied by a peelable tableau that arises from an application of Definition 13. At each stage where the leftmost nonempty column C of the current shape is an initial segment, the occurrence of C in Q is shown in bold:

$$D_1 = \begin{array}{cccc} \times & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \to \begin{array}{ccc} \times & \times & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{array} \to \begin{array}{cc} \times & \cdot \\ \cdot & \times \\ \times & \times \end{array} \to \begin{array}{cc} \times & \cdot \\ \times & \times \\ \cdot & \times \end{array} \to \begin{array}{c} \cdot \\ \times \\ \times \end{array} \to \begin{array}{c} \times \\ \cdot \\ \times \end{array} \to \begin{array}{c} \times \\ \times \\ \cdot \end{array} \to \varnothing$$

$$Q_1 = \begin{array}{cccc} \mathbf{1} & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{array} \quad \begin{array}{ccc} \mathbf{1} & 1 & 3 \\ \mathbf{2} & 2 & \\ 3 & & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & \\ 3 & \end{array} \quad \begin{array}{cc} \mathbf{1} & 2 \\ \mathbf{2} & \\ 3 & \end{array} \quad \begin{array}{c} 2 \\ 3 \end{array} \quad \begin{array}{c} 1 \\ 3 \end{array} \quad \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \quad \varnothing$$

Remark 15. Let D be a %-avoiding shape. Here is a recursive algorithm for producing the set of D-peelable tableaux.

If the first nonempty column C of D is an initial segment $[k]$, then assume by induction on $<_{ortho}$ that the set of $(D-C)$-peelable tableaux has already been computed. Let $\hat{Q}$ be a $(D-C)$-peelable tableau and X a skew standard tableau that labels from top to bottom, the cells of a vertical strip of cardinality $|C|$ that extends the shape of $\hat{Q}$. Compute the skew tableau $j_X(\hat{Q})$ by sliding $\hat{Q}$ to the southeast into the cells of X (in order from top to bottom, by definition). Since the jeu-de-taquin preserves descents, it follows that the standard tableau of vacated cells $v_X(\hat{Q}) = j^{\hat{Q}}(X)$ is the standard tableau (call it C) whose shape is a column of size k. Place the values 1 through k into these vacated cells, forming the tableau $C + j_X(\hat{Q})$. The set D-peelable tableaux is then given by all tableaux of this form that are column-strict, as $\hat{Q}$ ranges over the $(D-C)$-peelable tableaux and shape(X) ranges over the vertical strips of size k that extend shape$(\hat{Q})$. Note that $C + j_X(\hat{Q})$ is column-strict if and only if the value in the cell $(k+1, 1)$ is strictly greater than k.

If the first nonempty column of D is not an initial segment, then there is an i such that the ith row of D is properly contained in the $(i+1)$th (see the proof of Proposition 12). Assume by induction on $<_{ortho}$ that the set of $s_i(D)$-peelable tableaux has been computed and apply the plactic transposition s_i to each of these to obtain the set of D-peelable tableaux.

Example 16. Let us apply this algorithm to the shape D below. One first computes a saturated chain of shapes in the orthodontic poset from D down to $\varnothing$ by applications of Proposition 12.

$$D = \begin{array}{cc} \times & \cdot \\ \cdot & \times \\ \times & \times \end{array} \to \begin{array}{cc} \times & \cdot \\ \times & \times \\ \cdot & \times \end{array} \to \begin{array}{c} \cdot \\ \times \\ \times \end{array} \to \begin{array}{c} \times \\ \cdot \\ \times \end{array} \to \begin{array}{c} \times \\ \times \\ \cdot \end{array} \to \varnothing.$$

The set of peelable tableaux are constructed for each of the shapes in the chain, proceeding from the smallest to largest. It is easy to do this for the first three steps; the set of peelable tableaux for $\left(\begin{array}{c} \cdot \\ \times \\ \times \end{array}\right)$ is the singleton $\left\{\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right\}$. The peelable tableaux for the shape $\left(\begin{array}{cc} \times & \cdot \\ \times & \times \\ \cdot & \times \end{array}\right)$ are obtained by adjoining vertical strips of size 2 (denoted by the letters v_1 and v_2) in all possible ways to the previous set of tableaux,

$$\left\{ \begin{array}{ll} 2 & v_1 \\ 3 & v_2 \\ & \\ & \end{array}, \quad \begin{array}{ll} 2 & v_1 \\ 3 & \\ v_2 & \\ & \end{array}, \quad \begin{array}{l} 2 \\ 3 \\ v_1 \\ v_2 \end{array} \right\},$$

sliding these tableaux into the vertical strips from top to bottom,

$$\left\{ \begin{array}{ll} v_1 & 2 \\ v_2 & 3 \\ & \\ & \end{array}, \quad \begin{array}{ll} v_1 & 2 \\ v_2 & \\ 3 & \\ & \end{array}, \quad \begin{array}{l} v_1 \\ v_2 \\ 2 \\ 3 \end{array} \right\},$$

replacing v_1 and v_2 by 1 and 2 and discarding the tableaux that are not column-strict:

$$\left\{ \begin{array}{ll} 1 & 2 \\ 2 & 3 \\ & \\ & \end{array}, \quad \begin{array}{ll} 1 & 2 \\ 2 & \\ 3 & \\ & \end{array}, \quad \begin{array}{l} 1 \\ 2 \\ 2 \\ 3 \end{array} \leftarrow \text{discard} \right\}.$$

The two tableaux which were not discarded above are the peelable tableaux for the shape just to the right of D in the chain. The D-peelable tableaux are obtained by applying the plactic transposition s_2 to each of these two tableaux:

$$\left\{ \begin{array}{ll} 1 & 2 \\ 3 & 3, \\ & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \\ 3 & \end{array} \right\},$$

For aesthetic reasons we offer an alternate formulation for peelability.

Definition-Proposition 17. Let D be a %-avoiding shape. There is a well-defined operator τ_D given by orthodontic induction as follows:

1. $\tau_{\varnothing}$ is the identity operator.
2. If $D - C <_{ortho} D$ with $C = [k]$ then $\tau_D = \tau_{D-C}[k+1, r]$ (left juxtaposition by the word $[k+1, r]$ followed by τ_{D-C}).
3. If $(s_i, id)\, D <_{ortho} D$ then $\tau_D = \tau_{(s_i, id)\, D} s_i$, where s_i is the plactic transposition.

Proposition 18. *Let D be a %-avoiding shape. Then the word a is D-peelable if and only if $\tau_D(a)$ is Knuth equivalent to the unique $r \times c$ column-strict tableau with entries in $[c]$.*

Example 19. If D_1 is as in Example 10, then the operator τ_{D_1} is given by

$$\tau_{D_1}(a) = 3\ s_2 s_1\ 3\ s_2\ 3\ 32\ a.$$

One can check that applying the operator τ_{D_1} to any of the D_1-peelable tableaux from Example 14, produces a word that is Knuth equivalent to the tableau

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2. \\ 3 & 3 & 3 & 3 \end{matrix}$$

The main result of this section is Theorem 1, whose statement we recall here.

Theorem 1. *Let D be a %-avoiding shape. Then the Specht module Sp_D and Schur module $\mathcal{S}_D$ have the following decomposition into irreducible representations*:

$$\mathrm{Sp}_D \cong \bigoplus_{Q \text{ is } D\text{-peelable}} \mathrm{Sp}_{\mathrm{shape}(Q)} \tag{3.3.1}$$

$$\mathcal{S}_D \cong \bigoplus_{Q \text{ is } D\text{-peelable}} \mathcal{S}_{\mathrm{shape}(Q)}. \tag{3.3.2}$$

Theorem 1 will be deduced from the following combinatorial description of the decomposition of the character of a flagged Schur module into key polynomials. Let $\mathfrak{s}_D = \mathrm{char}(\mathcal{S}_D^{\mathrm{flag}})$.

THEOREM 20. *Let D be a %-avoiding shape. Then*

$$\mathfrak{s}_D = \sum_{D\text{-peelable tableaux } Q} \kappa_{\mathrm{content}(K_- Q)} \tag{3.3.3}$$

$$= \sum_{\substack{D\text{-peelable tableaux } Q \\ K_+ P \leqslant K_- Q}} x^{\mathrm{content}(P)} \tag{3.3.4}$$

$$= \sum_{\substack{\mathrm{rev}(a) \text{ is } D\text{-peelable} \\ i \text{ is } a\text{-compatible}}} x^{\mathrm{content}(i)}. \tag{3.3.5}$$

The equality of the expressions in Theorem 20 follows immediately from (2.2.7) and (2.2.8). Theorem 1 follows from Theorem 20 and the following result, conjectured in [26] for arbitrary shapes and proven by Magyar for %-avoiding shapes.

THEOREM 21 [19]. *Let D be a %-avoiding shape. Then*

$$\mathrm{char}(\mathscr{S}_D) = \pi_{w_0}\, \mathrm{char}(\mathscr{S}^{\mathrm{flag}}_D),$$

where w_0 is the longest element of $\sum_N$ and $\pi_{w_0} = \pi_{a_1}\pi_{a_2}\cdots\pi_{a_p}$, where $w_0 = s_{a_1}s_{a_2}\cdots s_{a_p}$ is a factorization of w_0 into a minimal number of adjacent transpositions s_i. In particular, for the composition $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$,

$$s_\lambda(x_1, x_2, ..., x_N) = \pi_{x_0} \kappa_\alpha,$$

where λ is the partition obtained by sorting the parts of α into weakly decreasing order.

Theorem 21 is not necessary to obtain Theorem 1 from Theorem 20. An alternate method is to apply the well-known special case of Theorem 21, where $D = D(\alpha)$ and exploit stability properties of the flagged expansion.

EXAMPLE 22. Example 14 computed the set of D_1-peelable tableaux for the shape D_1 in Example 10. Since these peelable tableaux have Ferrers shapes

$$\{(3, 2, 2), (3, 3, 1), (4, 2, 1), (4, 2, 1), (4, 3)\},$$

it follows from Theorem 1 that

$$\mathrm{Sp}_{D_1} \cong \mathrm{Sp}_{(3,2,2)} \oplus \mathrm{Sp}_{(3,3,1)} \oplus \mathrm{Sp}_{(4,2,1)} \oplus \mathrm{Sp}_{(4,2,1)} \oplus \mathrm{Sp}_{(4,3)}$$

$$\mathscr{S}_{D_1} \cong \mathscr{S}_{(3,2,2)} \oplus \mathscr{S}_{(3,3,1)} \oplus \mathscr{S}_{(4,2,1)} \oplus \mathscr{S}_{(4,2,1)} \oplus \mathscr{S}_{(4,3)}.$$

Using the algorithm from Section 2, one can compute the contents of the left keys of each of the D_1-peelable tableaux, obtaining the compositions

$$\{(3, 2, 2), (3, 3, 1), (4, 2, 1), (4, 1, 2), (4, 3)\}.$$

It follows from Theorem 20 that

$$\begin{aligned}\operatorname{char}(\mathscr{S}_{D_1}^{\text{flag}}) &= \kappa_{(3,2,2)} + \kappa_{(3,3,1)} + \kappa_{(4,2,1)} + \kappa_{(4,1,2)} + \kappa_{(4,3)} \\ &= x_1^3 x_2^2 x_3^2 + x_1^3 x_2^3 x_3^1 + x_1^4 x_2^2 x_3^1 + (x_1^4 x_2^2 x_3^1 + x_1^4 x_2^1 x_3^2) + x_1^4 x_2^3.\end{aligned}$$

The proof of (3.3.5) is based on Magyar's recurrence for the characters of the flagged Schur modules of %-avoiding shapes.

THEOREM 23 [20]. *The characters of the flagged Schur modules of %-avoiding shapes satisfy the following recurrence*:

S1. $\mathfrak{s}_{\varnothing} = 1$.

S2. *If* $D - C <_{ortho} D$ *with* $C = [k]$, *then*

$$\mathfrak{s}_D = x_1 x_2 \cdots x_k \mathfrak{s}_{D-C}.$$

S3. *If* $(s_i, id)\, D <_{ortho} D$, *then*

$$\mathfrak{s}_D = \pi_i \mathfrak{s}_{(s_i, id)\, D}.$$

EXAMPLE 24. Let D_1 be as in the preceding example. Consider the following saturated chain in the orthodontic poset from D_1 down to $\varnothing$:

$$D_1 = \begin{array}{cccc} \times & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \rightarrow \begin{array}{ccc} \times & \times & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{array} \rightarrow \begin{array}{cc} \times & \cdot \\ \cdot & \times \\ \times & \times \end{array} \rightarrow \begin{array}{cc} \times & \cdot \\ \times & \times \\ \cdot & \times \end{array} \rightarrow \begin{array}{c} \cdot \\ \times \\ \times \end{array} \rightarrow \begin{array}{c} \times \\ \cdot \\ \times \end{array} \rightarrow \begin{array}{c} \times \\ \times \\ \cdot \end{array} \rightarrow \varnothing$$

By Magyar's recurrence, $\mathfrak{s}_{D_1}$ is given by

$$\begin{aligned}\mathfrak{s}_{D_1} &= x_1 \cdot x_1 x_2 \pi_2(x_1 x_2 \pi_1 \pi_2(x_1 x_2)) \\ &= x_1 \cdot x_1 x_2 \pi_2(x_1 x_2 \pi_1(x_1 x_2 + x_1 x_3)) \\ &= x_1 \cdot x_1 x_2 \pi_2(x_1 x_2(x_1 x_2 + x_1 x_3 + x_2 x_3)) \\ &= x_1 \cdot x_1 x_2 \pi_2(x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3) \\ &= x_1 \cdot x_1 x_2(x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2) \\ &= x_1^4 x_2^3 + 2x_1^4 x_2^2 x_3 + x_1^3 x_2^3 x_3 + x_1^4 x_2 x_3^2 + x_1^3 x_2^2 x_3^2.\end{aligned}$$

This last answer agrees with the calculation from Example 22.

In light of the orthodontic induction, the recurrence in Theorem 23 uniquely defines the characters of flagged Schur modules of %-avoiding shapes.

The fact that the characters satisfy S1 is trivial, and there is a simple combinatorial proof (which we omit) that they satisfy S2. It is also easy to show that the polynomials obtained by following different saturated chains in the orthodontic poset from D down to $\varnothing$, must agree. It is *not* easy to prove combinatorially that the characters $\mathfrak{s}_D$ satisfy S3, a fact that is closely related to Theorem 21. Magyar's proof uses deep geometric results about the *configuration varieties* which he associates tp %-avoiding shapes.

4. SYMMETRIES OF D-PEELABLE TABLEAUX

The goal of this section is to define three symmetry operations %-avoiding shapes and on peelable tableaux, to restate Theorem 2, and to illustrate some of these notions with examples. All proofs are deferred to Appendix A2.

DEFINITION 25. Given a %-avoiding shape D that is contained in an $r \times c$ rectangular box, its *evacuation* $\mathrm{Evac}(D)$, *box-complement* $\mathrm{Boxcomp}(D)$, and *transpose* $\mathrm{Tr}(D)$ may be defined formally as follows:

$$\mathrm{Evac}(D) = \{(r+1-i, c+1-j) : (i, j) \in D\}$$

$$\mathrm{Boxcomp}(D) = \{(i, c+1-j) : (i, j) \notin D\}$$

$$\mathrm{Tr}(D) = \{(j, i) : (i, j) \in D\}.$$

$\mathrm{Evac}(D)$ is the 180° rotation of D within the $r \times c$ box, $\mathrm{Boxcomp}(D)$ is the left-to-right reflection of the set complement of D inside the $r \times c$ box, and $\mathrm{Tr}(D)$ is reflection of D across the diagonal $\{(i, i): i \in \mathbb{Z}_+\}$.

It is trivial to verify that

- all three maps preserve the %-avoiding property,
- all three maps are involutions, they satisfy the commutation relations specified by Theorem 2 and, hence, generate a dihedral group of order 8,
- $\mathrm{Evac}(D)$ and $\mathrm{Boxcomp}(D)$ are contained in the same $r \times c$ box as D is, while $\mathrm{Tr}(D)$ is contained in a $c \times r$ box.

EXAMPLE 26. Let D_1 be the shape from Example 10:

$$D_1 = \begin{array}{cccc} \times & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \qquad \mathrm{Evac}(D_1) = \begin{array}{cccc} \times & \times & \cdot & \cdot \\ \times & \cdot & \times & \cdot \\ \cdot & \times & \times & \times \end{array}$$

$$\mathrm{Boxcomp}(D_1) = \begin{array}{cccc} \times & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \qquad \mathrm{Tr}(D_1) = \begin{array}{ccc} \times & \cdot & \cdot \\ \times & \times & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{array}$$

On the level of Specht modules or Schur modules, each of these involutions has a known effect. Since $\mathrm{Evac}(D)$ is obtained from D by reversing the order of its rows and columns, the corresponding Specht (resp. Schur) modules are isomorphic as representations of $\sum_n$ (resp. $GL(N)$). Hence, Theorem 1 implies that there is a shape-preserving bijection between D-peelables and $\mathrm{Evac}(D)$-peelables.

It was conjectured by the authors [26] and proven by Magyar [19] that for arbitrary shapes D,

$$\mathscr{S}_{\mathrm{Boxcomp}(D)} \cong \det^{\otimes c} \otimes (\mathscr{S}_D)^*,$$

where det is the one-dimensional *determinant* representation of $GL(N)$, and $(\cdot)^*$ denotes the *contragredient* representation. Hence, Theorem 1 implies that there is a bijection between D-peelables and $\mathrm{Boxcomp}(D)$-peelables which sends each tableau of shape λ to a tableau of the Ferrers shape obtained by sorting the rows of the shape $\mathrm{Boxcomp}(\lambda)$.

It is easy to show that

$$\mathrm{Sp}_{\mathrm{Tr}(D)} \cong \mathrm{sgn} \otimes \mathrm{Sp}_D,$$

where *sgn* is the one-dimensional *sign* representation of $\sum_n$. Hence, Theorem 1 implies that there is a shape-transposing bijection between D-peelables and $\mathrm{Tr}(D)$-peelables.

The striking aspect of Theorem 2 is that each of these three bijections on peelables can be achieved by natural involutions which satisfy the same commutation properties as the corresponding operations on shapes.

Definition 27. Corresponding to Evac on shapes is the *evacuation* involution on column-strict tableaux. Given a column-strict tableau Q of Ferrers shape with entries in the interval $[r]$, define $\mathrm{Evac}(Q)$ to be the unique column strict tableau with entries in $[r]$ such that

$$\mathrm{shape}(\mathrm{Evac}(Q)|_{[i]}) = \mathrm{shape}(P(Q|_{[r+1-i,\, r]}))$$

for all $1 \leqslant i \leqslant r$, where $T|_I$ denotes the restriction of T to the subinterval I, that is, the skew column-strict tableau obtained by removing the entries of the (skew) column-strict tableau T that are not in the interval I.

It is well known that $\mathrm{Evac}(Q)$ can also be computed from Q by:

- rotating through $180°$,
- replacing each letter i by $r+1-i$,
- and sliding to Ferrers shape.

The result of the first two steps is a column-strict tableau called the *antitableau* $\mathrm{Anti}(Q)$ or *contretableau* of Q. Note that the value r, which does not appear in the notation $\mathrm{Anti}(Q)$ and $\mathrm{Evac}(Q)$, is implicit in the definition.

EXAMPLE 28. Let the shape D_1 and its peelable tableau Q_1 be as in Example 14. The evacuation $\mathrm{Evac}(Q_1)$ is computed in two ways. Let $r=3$ and $T_i = P(Q_1|_{[r+1-i,\, r]})$ for $1 \leqslant i \leqslant r$. The tableaux T_3, T_2, T_1, T_0 are given by

$$Q_1 = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{array} \qquad \begin{array}{ccc} 2 & 2 & 3 \\ 3 & & \end{array} \qquad \begin{array}{cc} 3 & 3 \end{array} \qquad \varnothing.$$

Thus $\mathrm{Evac}(Q_1)$ is defined by the sequence of partitions $() \subset (2) \subset (3,1) \subset (4,2,1)$. Alternatively, $\mathrm{Evac}(Q_1) = P(\mathrm{Anti}(Q_1))$:

$$\mathrm{Anti}(Q_1) = \begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 2 & 2 \\ 1 & 3 & 3 & 3 \end{array} \qquad \mathrm{Evac}(Q_1) = P(\mathrm{Anti}(Q_1)) = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & & \\ 3 & & & \end{array}.$$

DEFINITION 29. Corresponding to Boxcomp on shapes is the following well-known involution on tableaux (see, e.g., [31]). If Q is a column-strict tableau of Ferrers shape with entries in the interval $[r]$ and at most c columns, let $\mathrm{Boxcomp}(Q)$ be the column-strict tableau of Ferrers shape, whose jth column (viewed as a set) is the complement of the $(c+1-j)$th column of Q in $[r]$.

EXAMPLE 30. For D_1 and Q_1 as before, we have

$$Q_1 = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \end{matrix}, \qquad \mathrm{Boxcomp}(Q_1) = \begin{matrix} 1 & 2 & 3 & \cdot \\ 2 & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}.$$

If T is a filling of D, the usual definition of the transpose of T is the filling $\mathrm{tr}\, T$ of $\mathrm{Tr}\, D$ given by $(\mathrm{tr}\, T)(j, i) = T(i, j)$ for all cells $(i, j) \in D$. This naive notion of transpose is clearly unsuitable for column-strict tableaux.

DEFINITION-PROPOSITION 31. There is a well-defined shape-transposing map Tr_D from D-peelable tableaux of Ferrers shape to column-strict tableaux of Ferrers shape, defined by orthodontic induction as follows: Let Q be a D-peelable tableau of Ferrers shape,

Tr1. $\mathrm{Tr}_\varnothing(\varnothing) = \varnothing$.

Tr2. If the first nonempty column C of D (say the jth) is an initial segment $[k]$, then $\mathrm{Tr}_D\, Q$ is the unique column-strict tableau such that

a. $\mathrm{shape}(\mathrm{Tr}_D\, Q) = \mathrm{Tr}(\mathrm{shape}(Q))$.

b. $\mathrm{Tr}_D\, Q$ contains exactly k occurrences of its smallest entry j as an initial segment R of its first row, and $P((\mathrm{Tr}_D\, Q) - R) = \mathrm{Tr}_{D-C}\, P(Q - C)$.

Tr3. If the ith row of D is properly contained in the $(i+1)$th, then $\mathrm{Tr}_D\, Q = \mathrm{Tr}_{(s_i, id)\, D}\, s_i Q$.

If D is a skew shape, there is a well-known bijection L from the Littlewood–Richardson fillings T of shape D and content ν, to the D-peelable tableaux Q of Ferrers shape ν, such that the multiplicity of the letter i in the jth row of Q is equal to the multiplicity of the letter j in the ith row of T. It can be shown that the bijection $\mathrm{L}^{-1}\, \mathrm{Tr}\, \mathrm{L}$ coincides with a bijection of Hanlon and Sundaram [7] and White [34] between Littlewood–Richardson fillings for D and $\mathrm{Tr}(D)$. All of these bijections correspond to ordinary transposition when stated in the language of Zelevinsky's *pictures* [36].

EXAMPLE 32. The recursive rules in Definition 31 lead to the following algorithm for computing Tr on peelable tableaux. Let D_1 and Q_1 be as usual. The left two columns exhibit the D_1-peeling of Q_1 from before, and reading the right two columns from bottom to top shows the steps in the definition of Tr:

$$\begin{matrix} \times & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{matrix} \qquad \begin{matrix} \times & \cdot & \cdot \\ \times & \times & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{matrix} \qquad \begin{matrix} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \\ 4 & & \end{matrix}$$

$$\begin{array}{cccc} \cdot & \times & \times & \cdot \\ \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \qquad \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & & \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \times & \cdot \\ \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \times & \times \end{array} \qquad \begin{array}{cc} 1 & 3 \\ 2 & \\ 3 & \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \times & \cdot \\ \cdot & \cdot & \times & \times \\ \cdot & \cdot & \cdot & \times \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 2 & \\ 3 & \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \times \end{array} \qquad \begin{array}{c} 2 \\ 3 \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \times \end{array} \qquad \begin{array}{c} 1 \\ 3 \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{c} 1 \\ 2 \end{array}$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \varnothing$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \times & \times & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{array} \qquad \begin{array}{ccc} 2 & 2 & 4 \\ 3 & 3 & \\ 4 & & \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \times & \cdot & \times \\ \cdot & \times & \times \end{array} \qquad \begin{array}{ccc} 3 & 3 & 4 \\ 4 & & \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \times & \times & \cdot \\ \cdot & \times & \times \end{array} \qquad \begin{array}{ccc} 3 & 3 & 4 \\ 4 & & \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \times & \times \end{array} \qquad \begin{array}{cc} 4 & 4 \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \times & \cdot & \times \end{array} \qquad \begin{array}{cc} 4 & 4 \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \times & \times & \cdot \end{array} \qquad \begin{array}{cc} 4 & 4 \end{array}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \varnothing$$

THEOREM 2. *For each of the involutions* $i = \mathrm{Evac}$, Tr, $\mathrm{Boxcomp}$ *on the family of %-avoiding shapes D, the corresponding maps* $i = \mathrm{Evac}$, Tr, $\mathrm{Boxcomp}$ *on tableaux give involutive bijections from the D-peelable tableaux to the* $i(D)$*-peelable tableaux. Furthermore, these involutions on tableaux satisfy the same commutation relations as they do on shapes, namely,*

$$\text{Evac Tr} = \text{Tr Evac}$$

$$\text{Evac Boxcomp} = \text{Boxcomp Evac}$$

$$\text{Tr Boxcomp} = \text{Evac Boxcomp Tr}.$$

These involutions generate a dihedral group of order 8.

5. BRANCHING RULES, CORNER CELLS, AND HORIZONTAL STRIPS FOR NORTHWEST SHAPES

The goal of the section is to define corner cells and horizontal strips for northwest shapes, to restate Theorem 2, and to illustrate some of these notions with examples. All proofs are deferred to Appendix A4.

The definition of a corner cell of a northwest shape is facilitated by the *initial segment* partial order on subsets of $\mathbb{Z}_+$.

DEFINITION 33. Given two finite subsets R_1, R_2 of $\mathbb{Z}_+$, say that $R_1 <_{init} R_2$ if R_1 is a proper initial segment of R_2. Say that the rows of D are in *initial segment order* if whenever the i_1th row is a proper initial segment in the i_2th, then $i_1 < i_2$.

PROPOSITION 34. *Given any northwest shape D, there is a unique northwest shape D′ with rows in initial segment order, that is obtainable by reordering the rows of D. Furthermore, D′ may be obtained from D by a sequence of exchanges of adjacent rows which are comparable in* $<_{init}$.

EXAMPLE 35.

$$D = \begin{array}{cccc} \times & \times & \times & \\ \times & & & \\ & \times & \times & \times \\ & \times & \times & \\ & \times & & \\ & & \times & \end{array} \qquad D' = \begin{array}{cccc} \times & & & \\ \times & \times & \times & \\ & \times & & \\ & \times & \times & \\ & \times & \times & \times \\ & & \times & \end{array}$$

In light of the previous proposition and the fact that the isomorphism classes of the modules Sp_D and $\mathscr{S}_D$ depend only on the shape D up to row and column reorderings, it suffices to define corner cells and horizontal strips for northwest shapes with rows in initial segment order. It also turns out to be more convenient to define vertical strips first, and then transpose the answer.

Definition 36. Given a northwest shape D with rows in initial segment order, a *vertical strip* of D is a subset Y of its cells having the following three properties:

VS1. Y intersects each column of D in a final segment of that column (possibly empty).

VS2. Y contains at most one cell from each row of D.

VS3. This unique cell of Y in a given row of D is as far to the left as possible while preserving property VS1.

With the same conditions on D, a *corner cell* is a cell y which forms a vertical strip of cardinality 1. Clearly one can then define a *horizontal strip* of a northwest shape D whose *columns* are in initial segment order, to be the transpose of a vertical strip in $\mathrm{Tr}(D)$. We leave it to the reader to translate the defining properties VS1, VS2, VS3 into the defining properties HS1, HS2, HS3 of a horizontal strip.

In the familiar case of Ferrers and skew Ferrers shapes, if one reverses the order of the columns one obtains a northwest shape with rows in initial segment order, and this definition coincides with the usual notions of corner cell and vertical strip.

Example 37. For the shape D' in Example 35, the corner cells are the following:

$$
\begin{array}{cccc}
\times & & & \\
\otimes & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \times & \times & \times \\
 & & \times &
\end{array}
\qquad
\begin{array}{cccc}
\times & & & \\
\times & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \otimes & \times & \times \\
 & & \times &
\end{array}
\qquad
\begin{array}{cccc}
\times & & & \\
\times & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \times & \times & \times \\
 & & \otimes &
\end{array}
.
$$

Its vertical strips of size 2 are

$$
\begin{array}{cccc}
\otimes & & & \\
\otimes & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \times & \times & \times \\
 & & \times &
\end{array}
\quad
\begin{array}{cccc}
\times & & & \\
\otimes & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \otimes & \times & \times \\
 & & \times &
\end{array}
\quad
\begin{array}{cccc}
\times & & & \\
\otimes & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \times & \times & \times \\
 & & \otimes &
\end{array}
\quad
\begin{array}{cccc}
\times & & & \\
\times & \times & \times & \\
 & \times & & \\
 & \otimes & \times & \\
 & \otimes & \times & \times \\
 & & \times &
\end{array}
\quad
\begin{array}{cccc}
\times & & & \\
\times & \times & \times & \\
 & \times & & \\
 & \times & \times & \\
 & \otimes & \times & \times \\
 & & \otimes &
\end{array}
$$

We can now restate Theorem 3.

THEOREM 3. *Let D be a northwest shape of cardinality n with columns in initial segment order. Then we have an isomorphism of Σ_{n-1}-modules*

$$\operatorname{Res}^{\Sigma_n}_{\Sigma_{n-1}} \mathrm{Sp}_D \cong \bigoplus_{y \text{ a corner cell of } D} \mathrm{Sp}_{D-y}$$

and an isomorphism of $GL(N-1, \mathbb{F})$-modules

$$\operatorname{Res}^{GL(N, \mathbb{F})}_{GL(N-1, \mathbb{F})} \mathscr{S}_D \cong \bigoplus_{Y \text{ a horizontal strip of } D} \mathscr{S}_{D-Y}.$$

In the case of a Ferrer shape λ, after left-to-right reflection its columns will be in initial segment order and this theorem coincides with the usual branching rules for irreducibles of Σ_n and $GL(N, \mathbb{F})$ [27, Section 2.8].

EXAMPLE 38. Let D be the shape

$$\begin{array}{ccc} \times & \times & \times \\ \times & & \\ & \times & \\ & & \times \end{array}$$

whose columns are in initial segment order. The horizontal strips of D of various lengths are

$$\begin{array}{ccc} \times & \times & \times \\ \otimes & & \\ & \times & \\ & & \times \end{array} \qquad \begin{array}{ccc} \times & \times & \times \\ \times & & \\ & \otimes & \\ & & \times \end{array} \qquad \begin{array}{ccc} \times & \times & \otimes \\ \times & & \\ & \times & \\ & & \times \end{array}$$

$$\begin{array}{ccc} \times & \times & \times \\ \otimes & & \\ & \otimes & \\ & & \times \end{array} \qquad \begin{array}{ccc} \times & \times & \otimes \\ \otimes & & \\ & \times & \\ & & \times \end{array} \qquad \begin{array}{ccc} \times & \otimes & \otimes \\ \times & & \\ & \times & \\ & & \times \end{array} \qquad \begin{array}{ccc} \otimes & \otimes & \otimes \\ \times & & \\ & \times & \\ & & \times \end{array}$$

Theorem 1 says that Sp_D restricts as a representation of Σ_6 to the representation of Σ_5 given by the direct sum of the three Specht modules corresponding to the shapes obtained by removing the corner cells in the first three shapes above. It also says that $\mathscr{S}_D$ restricts as a representation of $GL(N, \mathbb{F})$ to the representation of $GL(N-1, \mathbb{F})$ given by the direct sum of the seven Schur modules corresponding to the shapes obtained by removing the horizontal strips shown above.

Combining Theorem 1 with the usual branching rules for $\sum_n$ and GL_N, and the fact that this decomposition of $\mathrm{Sp}_D, \mathscr{S}_D$ into irreducibles $\mathrm{Sp}_\lambda, \mathscr{S}_\lambda$ commutes with the operation Tr on shapes, Theorem 3 is equivalent to the following combinatorial result.

THEOREM 39. *Let D be northwest shape with rows in initial segment order. Then there is a bijection g*

$$\left\{ \begin{matrix} (Q, X)\text{: } Q \text{ is } D\text{-peelable} \\ X \text{ is a vertical strip of } \mathrm{shape}(Q) \end{matrix} \right\} \to \left\{ \begin{matrix} (Y, Q^-)\text{: } Y \text{ is a vertical strip of } D \\ Q^- \text{ is } (D-Y)\text{-peelable} \end{matrix} \right\}$$

with the property that $\mathrm{shape}(Q^-) = \mathrm{shape}(Q) - X$.

The bijection g has a natural definition. In the forward direction, given (Q, X), let Q^- be the tableau whose shape is $\mathrm{shape}(Q) - X$ resulting from successive reverse column insertions on Q at the cells of X in order from bottom to top, and let $i_1 i_2 \cdots i_{|X|}$ be the sequence of ejected values, which is strictly decreasing by Pieri's rule. It will be shown that this sequence gives the row indices of a unique vertical strip Y in D, thus defining (Y, Q^-). In the backward direction, given (Y, Q^-), let $Q = P(i_1 i_2 \cdots i_{|Y|} Q^-)$, where Y has row indices $i_1 > i_2 > \cdots > i_{|Y|}$, and let X be the vertical strip $\mathrm{shape}(Q)/\mathrm{shape}(Q^-)$.

EXAMPLE 40. The rows of the following shape D are in initial segment order.

$$D = \begin{matrix} \times & \times & & \\ \times & & \times & \\ \times & & & \times \end{matrix}$$

The D-peelable tableaux are

$$\left\{ \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{matrix}, \quad \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & & & \\ 3 & & & \end{matrix} \right\}.$$

Reverse column inserting each of these peelable tableaux at each of their vertical strips of size 2 and grouping the results according to the sequence of ejected values, we obtain

$$\begin{array}{ccc}
 & 1 & 3 \\
 & 2 & \\
 & 3 & \\
1 & & \\
2 & &
\end{array}$$

$$\begin{array}{ccc}
 & 1 & 2 \\
 & 2 & \\
 & 3 & \\
1 & & \\
3 & &
\end{array}
\qquad
\begin{array}{ccc}
 & 1 & 2 \\
 & 2 & 3 \\
 & & \\
1 & & \\
3 & &
\end{array}
\qquad
\begin{array}{cccc}
 & 1 & 2 & 3 \\
 & 2 & & \\
 & & & \\
1 & & & \\
3 & & &
\end{array}$$

$$\begin{array}{ccc}
 & 1 & 1 \\
 & 2 & \\
 & 3 & \\
2 & & \\
3 & &
\end{array}
\qquad
\begin{array}{ccc}
 & 1 & 1 \\
 & 2 & 3 \\
 & & \\
2 & & \\
3 & &
\end{array}
\qquad
\begin{array}{cccc}
 & 1 & 1 & 3 \\
 & 2 & & \\
 & & & \\
2 & & & \\
3 & & &
\end{array}
\qquad
\begin{array}{cccc}
 & 1 & 1 & 2 \\
 & 3 & & \\
 & & & \\
2 & & & \\
3 & & &
\end{array}
\qquad
\begin{array}{ccccc}
 & 1 & 1 & 2 & 3 \\
 & & & & \\
 & & & & \\
2 & & & & \\
3 & & & &
\end{array}$$

The above grouping shows that the resulting tableaux after column-ejection correspond to the peelables for the shapes obtained from D by removing all vertical strips of size 2:

$$\begin{array}{cccc}
\times & \cdot & & \\
\times & & \cdot & \\
\times & & & \times
\end{array}
\qquad
\begin{array}{cccc}
\times & \cdot & & \\
\times & & \times & \\
\cdot & & & \times
\end{array}
\qquad
\begin{array}{cccc}
\times & \times & & \\
\cdot & & \times & \\
\cdot & & & \times
\end{array}$$

Remark 41. One might ask for a similar branching rule for all %-avoiding shapes; however, the following example exhibits an obstacle. Consider the shape

$$D = \begin{array}{ccc}
\times & \times & \\
\times & & \times \\
 & \times & \times
\end{array}.$$

It is easy to see that all of the Specht modules Sp_{D-y} for any cell y of D are isomorphic. Theorem 1 implies that

$$\mathrm{Sp}_{D-y} \cong \mathrm{Sp}_{(2,2,1)} \oplus \mathrm{Sp}_{(3,1,1)} \oplus \mathrm{Sp}_{(3,2)}$$

as an $\sum_5$-representation. However, one can also use Theorem 1 and the usual $\sum_n$-branching rule to calculate that

$$\mathrm{Res}_{\sum_{n-1}}^{\sum_n} \mathrm{Sp}_D \cong \mathrm{Sp}_{(2,2,1)}^{\oplus 3} \oplus \mathrm{Sp}_{(3,1,1)}^{\oplus 2} \oplus \mathrm{Sp}_{(3,2)}^{\oplus 3}$$

as an $\sum_5$-representation. Therefore, it is impossible to have a branching rule of the form

$$\mathrm{Res}_{\sum_{n-1}}^{\sum_n} \mathrm{Sp}_D \cong \bigoplus_{y \text{ a corner cell of } D} \mathrm{Sp}_{D-y}$$

that is valid for all %-avoiding shapes D, no matter how one defines "corner cell."

6. REMARKS AND OPEN PROBLEMS

6.1. *Bases*

Theorems 1 and 20 suggest some explicit tableaux-like bases for the various modules Sp_D, $\mathscr{S}_D$, and $\mathscr{S}_D^{\mathrm{flag}}$ when D is %-avoiding. It is well known that when D is a skew shape, the set of skew tableaux of shape D index a basis for the skew Schur module S_D. It is also well known that the column-strictness of a filling T of a skew shape D is determined by the recording tableau of its reading word [35]. So when D is skew, bases of the above modules are indexed by fillings such that the recording tableau of the reading word has a property depending only on D. Such bases have been found for shapes of compositions, column-convex shapes, and diagrams of permutations [23–25].

The following definition is the appropriate modification of these constructions for the definition of the (flagged) Schur module given here. Let T be a filling of D whose columns strictly increase from top to bottom. Let fillingword(T) be the reverse of the reading word of T given by reading along columns from top to bottom, starting with the leftmost and proceeding to the right. Let $Q_*(T) = \mathrm{std}^{-1}(\mathrm{tr}\, Q(\text{fillingword}(T)), \beta)$, where β_j is the number of cells in the jth column of D.

Let $\mathscr{B}_D^{\mathrm{flag}}$ be the set of fillings T of D such that $Q_*(T)$ is $\mathrm{Tr}\, D$-peelable and $K_+ P(\text{fillingword}(T)) \leqslant K_- \mathrm{Tr}_{\mathrm{Tr}\, D}\, Q_*(T)$.

Let $\mathscr{B}_D^N$ be the set of fillings T of D such that $Q_*(T)$ is $\mathrm{Tr}\, D$-peelable and none of the entries of T exceed N.

Finally, let $\mathscr{B}_D^{std}$ be the set of fillings T of D such that $Q_*(T)$ is Tr D-peelable and T is a bijection $D \to [|D|]$.

Conjecture 42. $\{\Delta_T\}_{T \in \mathscr{B}_D^{std}}$, $\{\Delta_T\}_{T \in \mathscr{B}_D^N}$, and $\{\Delta_T\}_{T \in \mathscr{B}_D^{\mathrm{flag}}}$ are bases for the modules Sp_D, $\mathscr{S}_D$, and $\mathscr{S}_D^{\mathrm{flag}}$, respectively, valid over any base ring.

Independently Magyar defined the following set of fillings $\mathcal{M}_D$ (using a particular saturated chain in the orthodontic poset) and conjectured that $\{\Delta_T: T \in \mathcal{M}_D\}$ is a basis for $\mathcal{S}_D^{\text{flag}}$ [20].

DEFINITION 43. Let $\mathcal{M}_D$ be the set of fillings of D given as follows:

1. If $D = \varnothing$, let $\mathcal{M}_D = \{\varnothing\}$.

2. If $D - C <_{ortho} D$, let $\mathcal{M}_D$ consist of the fillings obtained by adjoining the column C to the left of each of the fillings in $\mathcal{M}_{D-C}$.

3. If $(s_i, id)\, D <_{ortho} D$: For $T \in \mathcal{M}_{(s_i,\, id)\, D}$, let $\pi_i(T)$ denote the set of fillings T' of D such that $\text{fillingword}(T') = f_i^p(\text{fillingword}(T))$ for some $0 \leqslant p \leqslant k$, where k is the number of i-unpaired i's in fillingword(T). Let $\mathcal{M}_D = \bigcup_{T \in \mathcal{M}_{(s_i,\, id)\, D}} \pi_i(T)$.

It will be shown that $\mathcal{M}_D$ has the intrinsic description given by $\mathcal{B}_D^{\text{flag}}$; this also shows that $\mathcal{M}_D$ is well-defined.

THEOREM 44. $\mathcal{M}_D = \mathcal{B}_D^{\text{flag}}$.

6.2. *Other Base Rings*

The constructions of Specht, Schur, and flagged Schur modules given in Section 2 are defined over any commutative ring R. Our definition of the Schur module is isomorphic to the functorial construction given in [1]. For %-avoiding shapes, it also coincides with a geometric construction of Magyar [19]. The flagged Schur module has similar algebraic and geometric descriptions. The Specht module can be defined simply as the (1^n)-weight subspace of the Schur module.

Let G_R and B_R denote the algebraic groups given by the general linear group $GL(N)$ and the torel subgroup of lower triangular matrices, with entries in R. To state results that hold for arbitrary R, one must work not in the category of G and B-modules, but rather in the category of $R[G]$ and $R[B]$-comodules, where $R[G]$ and $R[B]$ are the Hopf algebras of regular functions $G \to R$ and $B \to R$, respectively. Let R^N be the free R-module of rank N and let $\mathcal{S}_D(R^N)$ and $\mathcal{S}_D^{\text{flag}}(R^N)$ denote the Schur and flagged Schur modules of shape D over R. One has the following "universal freeness" result

THEOREM 45 [19, 32]. *Let D be a %-avoiding shape. For any commutative ring R, there are isomorphisms*

$$\mathcal{S}_D(R^N) \cong R \otimes_{\mathbb{Z}} \mathcal{S}_D(\mathbb{Z}^N)$$

$$\mathcal{S}_D^{\text{flag}}(R^N) \cong R \otimes_{\mathbb{Z}} \mathcal{S}_D^{\text{flag}}(\mathbb{Z}^N).$$

In particular, as R-modules, $\mathscr{S}_D(R^N)$ and $\mathscr{S}_D^{\mathrm{flag}}(R^N)$ are free, and their ranks are independent of R.

Next we state the characteristic-free version of Theorem 1. Say that a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$ of the $R[G]$ (resp. $R[B]$) comodule M is *good* (resp. *excellent*) if each of its successive quotients V_i/V_{i-1} is isomorphic to a Schur module of partition shape (resp. to a Demazure module). Let $R[\sum_n]$ be the ordinary group ring. A *Specht series* is a filtration of an $R[\sum_n]$-module whose successive quotients are isomorphic to Specht modules of partition shape. A Specht series for Sp_D may be obtained from a good filtration of $\mathscr{S}_D$ by taking (1^n)-weight spaces (when $n \leqslant N$).

For %-avoiding shapes, the Schur and flagged Schur modules have "universal" good and excellent filtrations, respectively.

THEOREM 46 [19, 32]. *Let D be a %-avoiding shape. Then $\mathscr{S}_D(R^N)$ has a good filtration and $\mathscr{S}_D^{\mathrm{flag}}(R^N)$ has an excellent filtration. Furthermore, these filtrations may be obtained by applying $R \otimes_{\mathbb{Z}} \cdot$ to the corresponding filtrations over $\mathbb{Z}$.*

It follows from linear independence of the Schur polynomials that if $\{V_i\}$ is any good filtration of $\mathscr{S}_D$ over any base ring R, then the number of indices i for which $V_i/V_{i-1} \cong S_\lambda(R^N)$, is equal to the number of D-peelable tableaux of shape λ. A similar statement holds for the flagged Schur module due to the linear independence of the key polynomials.

In [8, 25], explicit Specht series were obtained over arbitrary fields for skew shapes D and row-convex shapes D, respectively; the reason why we say row-convex (contrary to the title of [25]) is due to the fact that the specific Specht module construction used in [25] for a shape D is isomorphic to the module $\mathrm{Sp}_{\mathrm{Tr}(D)}$. The method of [8] is to produce a short chain complex

$$0 \to \mathrm{Sp}_{D^K} \to \mathrm{Sp}_D \to \mathrm{SP}_{D^I} \to 0$$

for every shape D in a family of shapes $\mathscr{C}$ that contains all skew shapes, such that D^K and D^I lie in $\mathscr{C}$ and are closer in some sense to Ferrers shapes. By induction the modules Sp_{D_K} and Sp_{D_I} have Specht series, which can be pieced together to give a chain complex for Sp_D. Then a dimension-counting argument is given to show that this chain complex is exact.

This method is adopted in [25] to obtain explicit Specht series for row-convex shapes. Using Theorems 45 and 46 together with the methods of [8] and [25], explicit good filtrations are obtained for the Schur modules of row-convex shapes [29] over an arbitrary base ring.

Conjecture 47. For any %-avoiding shape D, the method of [8] can be adapted to produce an explicit Specht series for Sp_D, a good filtration for

$\mathcal{S}_D$, and an excellent filtration for $\mathcal{S}_D^{\text{flag}}$, whose successive quotients are isomorphic to $\{\text{Sp}_{\text{shape}(Q)}\}$, $\{\mathcal{S}_{\text{shape}(Q)}\}$, and $\{\mathcal{S}^{\text{flag}}_{\text{content}(K\_Q)}\}$, respectively, as Q runs over the set of D-peelable tableaux.

6.3. *Branching*

In [10], over arbitrary fields Kraskiewicz proves a branching rule for Specht modules for shapes D which are *inversion diagrams* $I(w)$ of permutations w [10]. He defines the notion of a *descent cell* y in the shape $D = I(w)$ and exhibits a filtration of $\sum_{n_1}$-modules,

$$0 = V_0 \subseteq V_1 \subseteq \cdot V_{m-1} \subseteq V_m = \text{Res}^{\Sigma_n}_{\Sigma_{n-1}}(\text{Sp}_{I(w)}),$$

such that the successive quotients $\{V_i/V_{i-1}\}_{i=1}^m$ are isomorphic (in some order) to the modules

$$\{\text{Sp}_{I(w)-y}\}_{y \text{ a descent cell of } I(w)}.$$

Conjecture 48. For any northwest shape D, the method of [10] can be adapted to produce a $\sum_{n-1}$-filtration of $\text{Res}^{\Sigma_n}_{\Sigma_{n-1}}(\text{Sp}_D)$ whose successive quotients are isomorphic to $\{\text{Sp}_{D-y}\}$ as y runs over the corner cells of D, and also a $GL(N-1, \mathbb{F})$-filtration of $\text{Res}^{GL(N, \mathbb{F})}_{GL(N-1, \mathbb{F})}(\mathcal{S}_D)$ whose successive quotients are isomorphic to $\{\mathcal{S}_{D-Y}\}$ as Y runs over the horizontal strips of D.

APPENDIX A: PEELABLE TABLEAUX

Properties of the orthodontic poset and peelable tableaux are developed and the criterion for peelability given in Proposition 18 is proven.

A.1. *The Orthodontic Poset*

An *inversion* of a subset $C \subseteq [r]$ is a pair (i, j) with $1 \leqslant i < j \leqslant r$ such that $i \notin C$ but $j \in C$. An inversion of a shape is, by definition, an inversion of one of its columns. Denote by $\text{Inv}(D)$ the set of inversions of the shape D. Define the quantity

$$m(D) = \text{the number of nonempty columns of } D + |\text{Inv}(D)|.$$

We state and prove a more detailed version of Proposition 12.

PROPOSITION 49. *D is %-avoiding if and only if $\emptyset \leqslant_{ortho} D$. Furthermore, if D is %-avoiding then there is a permutation $w \in \sum_r$ such that*

$$\text{Inv}(D) = \text{Inv}(w) = \{(i, j): i < j \text{ and } w(i) > w(j)\}.$$

Proof. Suppose $D' <_{ortho} D$ is a covering relation between two shapes D' and D. We wish to show that $m(D') = m(D) - 1$. Suppose first that $D' = D - C <_{ortho} D$. It is easy to see that $D - C$ and D have the same set of inversions but $D - C$ has one less nonempty column, so that $m(D - C) = m(D) - 1$. Otherwise $D' = (s_i, id)\, D <_{ortho} D$. Then $(s_i, id)\, D$ and D have the same number of nonempty columns. Note that $(i, i+1) \in \mathrm{Inv}(D) - \mathrm{Inv}((s_i, id)\, D)$ since the ith row of D is properly contained in the $(i+1)$th. This proper containment of rows also shows that the map

$$\mathrm{Inv}(D) - \{(i, i+1)\} \to \mathrm{Inv}((s_i, id)\, D)$$

$$(p, q) \mapsto (s_i p, s_i q)$$

is a bijection. It follows that $(s_i, id)\, D$ has one less inversion than D does, so that $m((s_i, id)\, D) = m(D) - 1$.

Therefore $\leqslant_{ortho}$ gives the set of all shapes the structure of a graded poset using the function $m(D)$. By Remark 11 and the fact that the empty shape is %-avoiding, it follows that the family of %-avoiding shapes contains the connected component of the empty shape in the $\leqslant_{ortho}$ poset.

For the rest of the assertions, it remains to show by induction on $m(D)$ that if D is %-avoiding then $\varnothing \leqslant_{ortho} D$ and there is a permutation $w \in \sum_r$ such that $\mathrm{Inv}(D) = \mathrm{Inv}(w)$. Note that $m(D) = 0$ if and only if $D = \varnothing$, so that $\varnothing \leqslant_{ortho} D$ and $\mathrm{Inv}(D) = \mathrm{Inv}(id)$. So suppose that $m(D) > 0$. Suppose first that the leftmost nonempty column C of D is an initial segment. Then since $m(D - C) = m(D) - 1$, by induction we have $\varnothing \leqslant_{ortho} D - C <_{ortho} D$ and there is a permutation $v \in \sum_r$ such that $\mathrm{Inv}(D) = \mathrm{Inv}(D - C) = \mathrm{Inv}(v)$. Otherwise C is not an initial segment. Clearly C must have an inversion of the form $(i, i+1)$. By the %-avoiding property applied to the ith and $(i+1)$th rows and the leftmost nonempty column and any other column of D, it follows that the ith row of D is properly contained in the $(i+1)$th. This means that $(s_i, id)\, D <_{ortho} D$. It has been shown that $m((s_i, id)\, D) = m(D) - 1$, so that by induction $\varnothing \leqslant_{ortho} (s_i, id)\, D <_{ortho} D$ and there is a permutation $v \in \sum_r$ such that $\mathrm{Inv}((s_i, id)\, D) = \mathrm{Inv}(v)$. Moreover, $v(i) < v(i+1)$ since the ith row of $(s_i, id)\, D$ properly contains the $(i+1)$th. It follows that the map $\mathrm{Inv}(vs_i) - \{(i, i+1)\} \mapsto \mathrm{Inv}(v)$ given by $(p, q) \mapsto (s_i p, s_i q)$ is a bijection, and that $\mathrm{Inv}(vs_i) = \mathrm{Inv}(D)$. ∎

A.2. *Well-Definedness of Peelability*

Here is the proof that peelability is well defined (see Definition-Proposition 13).

Proof. The proof proceeds by orthodontic induction. It must be shown that for any pair of different covering relations of the form $D' <_{ortho} D$, the condition of D-peelability given by these two relations is consistent. Let Q

be a column strict tableau of partition shape and let C be the first nonempty column of D.

Case 1. $D-C <_{ortho} D$ and $(s_i, id)\, D <_{ortho} D$. Let $C=[k]$. Since $k \in C$ but $k+1 \notin C$ and the ith row of D is properly contained in the $(i+1)$th it follows that $k \neq i$. Suppose first that the first column of Q contains C. By the definition of the plactic action of s_i, the only positions in Q and $s_i Q$ that could possibly differ are those that contain i-unpaired letters. If $k<i$ then C does not contain any letters i or $i+1$. If $k>i$ then C contains the subword $(i+1)i$ of i-paired letters. Either way C does not contain any i-unpaired letter. Therefore the first column of $s_i Q$ contains C. The same argument shows that if the first column of $s_i Q$ contains C then so does the first column of Q. Thus we may assume without loss of generality that the first columns of Q and $s_i Q$ both contain C. Note that columnword$(Q)=u[k]\,v$, where u is the portion of the first column of Q that lies below C, and $v=$ columnword(T), where T is the tableau obtained from Q by removing its first column. Let $s_i(uv)=u'v'$, where u and u' have the same length (so v and v' do as well). Then $s_i(u[k]\,v)=u'[k]\,v'$. It follows that $P((s_i Q)-C) = P(s_i(Q-C)) = s_i P(Q-C)$. By induction, $P(Q-C)$ is $(D-C)$-peelable, if and only if $s_i P(Q-C)$ is $(s_i, id)(D-C)$-peelable, if and only if $P((s_i Q)-C)$ is $[((s_i, id)\, D)-C]$-peelable, if and only if $s_i Q$ is $(s_i, id)\, D$-peelable.

Case 2. $(s_i, id)\, D <_{ortho} D$ and $(s_j, id)\, D <_{ortho} D$ with $i<j$

Case 2a. $j>i+1$. It is clear that

$$(s_i s_j, id)\, D = (s_j s_i, id)\, D$$

$$(s_i s_j, id)\, Q = (s_j s_i, id)\, Q$$

$$(s_i s_j, id)\, D <_{ortho} (s_j, id)\, D <_{ortho} D$$

$$(s_j s_i, id)\, D <_{ortho} (s_i, id)\, D <_{ortho} D.$$

By induction, $s_i Q$ is $(s_i, id)\, D$-peelable if and only if $s_j s_i Q$ is $(s_j s_i, id)$ D-peelable, if and only if $s_i s_j Q$ is $(s_i s_j, id)\, D$-peelable, if and only if $s_j Q$ is $(s_j, id)\, D$-peelable.

Case 2b. $j=i+1$. We have $(s_i s_j s_i, id)\, D = (s_j s_i s_j, id)\, D$ and $s_i s_j s_i Q = s_j s_i s_j Q$ by the fact that the plactic transpositions define a group action. We are assuming that $(s_i, id)\, D <_{ortho} D$ and $(s_j, id)\, D <_{ortho} D$, so that the ith row of D is properly contained in the jth, which is properly contained in the $(j+1)$th. From this it follows that

$$(s_i s_j s_i, id)\, D <_{ortho} (s_j s_i, id)\, D <_{ortho} (s_i, id)\, D <_{ortho} D$$

$$(s_j s_i s_j, id)\, D <_{ortho} (s_i s_j, id)\, D <_{ortho} (s_j, id)\, D <_{ortho} D.$$

By induction, s_iQ is $(s_i, id)\,D$-peelable, if and only if s_js_iQ is (s_js_i, id) D-peelable, if and only if $s_is_js_iQ$ is $(s_is_js_i, id)\,D$-peelable, if and only if $s_js_is_jQ$ is $(s_js_is_j, id)\,D$-peelable, if and only if s_is_jQ is $(s_is_j, id)\,D$-peelable, if and only if s_jQ is $(s_j, id)\,D$-peelable. ∎

Remark 50. Here are a few obvious facts about the content and shape of peelable tableaux.

1. The content of any D-peelable tableau or word agrees with the composition whose ith component is the number of cells in the ith row of D for all i. In particular, if Q is a D-peelable tableau of partition shape and D has at most r nonempty rows, then Q has at most r rows.

2. If Q is a D-peelable tableau of partition shape and D has at most c nonempty columns, then so does Q. Here is a proof that proceeds by orthodontic induction. In the case $D-C <_{ortho} D$, the shape $D-C$ has one fewer column than D. Since the jeu-de-taquin is descent-preserving it follows that the shapes of $P(Q-C)$ and Q differ by a vertical strip. Thus $P(Q-C)$ has at most one fewer column than Q does. When $(s_i, id)\,D <_{ortho} D$, exchanging rows of the shape does not change the number of columns, and $\text{shape}(s_iQ) = \text{shape}(Q)$.

Shapes that differ by the exchange of nested adjacent columns have the same peelable tableaux!

Lemma 51. *Suppose that the jth column of D is properly contained in the $(j+1)$th. Then the notions of D-peelability and (id, s_j) D-peelability are equivalent.*

Proof. The proof proceeds by orthodontic induction. Without loss of generality assume that the first column C of D is nonempty. Let D be %-avoiding. The result is trivial when $D=\varnothing$:

Case 1. $(s_i, id)\,D <_{ortho} D$. Note that exchanges of rows and exchanges of columns commute. By induction, Q is D-peelable if and only if s_iQ is $(s_i, id)\,D$-peelable, if and only if s_iQ is $(id, s_j)(s_i, id)\,D = (s_i, id)(id, s_j)$ D-peelable, if and only if Q is $(id, s_j)\,D$-peelable.

Case 2. $D-C <_{ortho} D$. Let $C=[k]$.

Case 2a. $j>1$. The first column of $(id, s_j)\,D$ is also C, and (id, s_j) $(D-C) = ((id, s_j)\,D) - C$. Without loss of generality we may assume that the first column of Q contains C. Then by induction, Q is D-peelable if and only if $P(Q-C)$ is $(D-C)$-peelable, if and only if $P(Q-C)$ is (id, s_j) $(D-C) = ((id, s_j)\,D) - C]$-peelable, if and only if Q is $(id, s_j)\,D$-peelable.

Case 2b. $j=1$. Let C' be the second column of D.

Case 2b(i). C' is not an initial segment. Then it has an inversion $(i, i+1)$. Note that $i>k$ since $C \subset C'$. By the %-avoiding property of D it follows that the ith row of D is properly contained in the $(i+1)$th, and we are done by Case 1.

Case 2b(ii). C' is an initial segment (say $[l]$, with $l>k$). We have $D-C-C' = ((id, s_1)\, D) - C' - C$. Consider the chains of shapes $D-C-C' <_{ortho} D-C <_{ortho} D$ and $((id, s_1)\, D) - C' - C <_{ortho} ((id, s_1)\, D) - C' <_{ortho} (id, s_1)\, D$. By staring at various jeux-de-taquin, it can be shown that the following are equivalent:

(a) Q is D-peelable.

(b) The first column of Q contains C and $P(Q-C)$ is $(D-C)$-peelable.

(c) The first column of Q contains C, the first column of $P(Q-C)$ contains C', and $P(P(Q-C)-C')$ is $[(D-C)-C']$-peelable.

(d) The first and second columns of Q contain C' and C, respectively (denote by $C'+C$ this two-column subtableau), and $P(Q-(C'+C))$ is $[D-C-C']$-peelable.

(e) The first column of Q contains C', the first column of $P(Q-C')$ contains C, and $P(P(Q-C')-C)$ is $[(((id, s_1)\, D)-C')-C]$-peelable.

(f) The first column of Q contains C' and $P(Q-C')$ is $[((id, s_1)\, D)-C']$-peelable.

(g) Q is $(id, s_1)\, D$-peelable.

Moreover, $P(P(Q-C)-C') = P(Q-(C'+C)) = P(P(Q-C')-C)$. ∎

A.3. *Criterion for peelability*

In this subsection Proposition 18 is proven. To prove that τ_D is well-defined one must consider the same cases as in the proof of Definition-Proposition 13 (see Subsection A.2). Case 2 is resolved in exactly the same manner, and Case 1 is settled by the second item in the following remark.

Remark 52. These facts will be used often and without mention in the proofs that follow:

1. [14] Let u and v be strictly decreasing words with u weakly longer than v. Then $uv \sim vu$ if and only if u contains v when both are viewed as subsets.

2. If u is a strictly decreasing word containing both i and $i+1$ (or neither) then for any word v, $s_i(uv) = us_iv$.

The following simple observation is the basis of the peelability criterion of Proposition 18 and proves useful in proving the three symmetry bijections for peelables.

Proposition 53. *Let b be a word in the alphabet $[r]$ and $1 \leqslant k \leqslant r$. The following are equivalent*:

1. *b contains the subword $[k]$.*
2. *The first column of $P(b)$ contains $[k]$.*
3. *The first column of $P([k+1, r]b)$ equals $[r]$.*

Furthermore, if the above hold, then $P([k+1, r]\, Q)$ is the tableau obtained by adjoining a full column $[r]$ to the left of $P(Q-[k])$, where $Q = P(b)$.

Proof. For the equivalence of 1 and 2, in light of the fact that $P(b|_I) = P(b)|_I$ for initial subintervals I, we may assume that all of the letters of b are in the interval $[k]$. Then b contains the subword $[k]$ if and only if b contains a strictly decreasing subword of length k, if and only if $P(b)$ has k rows [28], if and only if the first column of $P(b)$ is given by $[k]$. For the equivalence of 2 and 3, let $Q = P(b)$. By two applications of the equivalence of 1 and 2, it follows that the first column of Q contains $[k]$ if and only if the word of Q contains the subword $[k]$, if and only if $[k+1, r]\, Q$ contains the subword $[r]$, if and only if the first column of $P([k+1, r]\, Q)$ is $[r]$.

Lastly, let T be the tableau obtained by removing the first column of Q, and c the decreasing word given by the portion of the first column of Q that is not in $[k]$. We have

$$\begin{aligned}[k+1, r]\, Q &= [k+1, r]\, c[k]\, T \sim [r]\, cT \sim [r]\, P(cT) \\ &= [r]\, P(Q-[k]). \quad \blacksquare\end{aligned}$$

Proof of Proposition 18. We show that a is peelable if and only if $P(\tau_D(a)) = \mathrm{key}((c^r))$. The proof proceeds by orthodontic induction. One case is $(s_i, id)\, D <_{ortho} D$. Then a is D-peelable if and only if $s_i a$ is $(s_i, id)\, D$-peelable, if and only if (by induction) $\tau_{(s_i,\, id)\, D}(s_i a) = \mathrm{key}((c^r))$, if and only if $\tau_D(a) = \mathrm{key}((c^r))$.

The remaining case is $(D-C) <_{ortho} D$. Let $C = [k]$. Suppose first that Q is D-peelable. Then the first column of Q contains C and $P(Q-C)$ is $(D-C)$-peelable. By Proposition 53 and induction, $P([k+1, r]\, Q) = P([r]\, P(Q-C))$ and $\tau_{D-C} P(Q-C) = \mathrm{key}((c-1)^r)$. It follows that

$$\begin{aligned}P(\tau_D Q) &= P(\tau_{D-C}[k+1, r]\, Q) = P(\tau_{D-C}[r]\, P(Q-C)) \\ &= P([r]\, \tau_{D-C} P(Q-C)) = P([r]\, \mathrm{key}((c-1)^r)) = \mathrm{key}((c^r)).\end{aligned}$$

For the converse, suppose Q is not D-peelable. The first possibility is that the first column of Q does not contain C. By Proposition 53 the first column of $T = P([k+1, r]\, Q)$ does not contain $[r]$, so that T has strictly less than r rows. Consider the shape of the tableau $P(\tau_{D-C}(T))$. A plactic transposition does not affect the shape of a tableau, and column insertion of a strictly decreasing word adjoins a vertical strip to the shape of the tableau by Pieri's rule. The operator τ_{D-C} has at most $c-1$ of these strictly decreasing words. Therefore shape$(P(\tau_{D-C}(T)))$ is obtained from shape(T) by adjoining at most $(c-1)$ vertical strips. However, the bottom row of the skew shape (c^r)/shape(T) has c cells, so that the shapes of $P(\tau_D(Q)) = P(\tau_{D-C}(T))$ and key$((c^r))$ can't be the same. The second possibility is that the first column of Q contains C but $P(Q-C)$ is not $(D-C)$-peelable. By induction $\tau_{D-C}(P(Q-C)) \neq \mathrm{key}(((c-1)^r)$. Since $\tau_{D-C}(P(Q-C))$ has content $((c-1)^r)$, it follows that its shape must strictly dominate the partition $((c-1)^r)$. By Proposition 53,

$$\begin{aligned} P(\tau_D(Q)) &= P(\tau_{D-C}[k+1, r]\, Q) \\ &= P(\tau_{D-C}[r]\; P(Q-C)) = P([r]\; \tau_{D-C}(P(Q-C))), \end{aligned}$$

whose shape is obtained from $P(\tau_{D-C}(P(Q-C)))$ by adding a column of length r. Thus the shapes of $P(\tau_D(Q))$ and key$((c^r))$ disagree. ∎

APPENDIX B: THE SYMMETRIES OF %-AVOIDING SHAPES

The proof of Theorem 2 is given here. This is preceded by a discussion of the four basic inductions on D-peelable tableaux, which correspond to removing the first row, last row, first column, or last column of the shape D.

B.1. *The Four Inductions on Peelables*

One of these (namely "west-hat" or wh) is essentially given in the definition of a peelable tableau in the case that the first column C of the shape D is an initial segment. This operation removes the column C from the shape D and reduces the D-peelable tableau Q to the $(D-C)$-peelable tableau $P(Q-C)$.

Let D be a %-avoiding shape, embedded in the $r \times c$ rectangle. Define the operators nh ("north-hat"), sh ("south-hat"), wh ("west-hat"), and eh ("east-hat") on %-avoiding shapes and their peelable tableaux as follows. The shapes nh D, sh D, wh D, and eh D are given by removing all the cells from the first row, last row, first column, and last column of D, respectively. It is easy to check that all of these shapes are %-avoiding. Let Q be a D-peelable tableau of partition shape. Let nh Q (resp. sh Q) be the column-strict tableau of partition shape that is Knuth equivalent to the tableau obtained from Q by removing every letter 1 (resp. r):

$$\mathrm{nh}\; Q = P(Q|_{[2,r]})$$

$$\mathrm{sh}\; Q = Q|_{[r-1]}.$$

It will be shown below that nh Q is (nh Q)-peelable and sh Q is (sh D)-peelable. Let us now define wh Q. Let C_w be the first column of D and $u_w \in \sum_r$ the shortest permutation such that $u_w C_w$ is an initial segment. Define

$$\text{wh } Q = u_w^{-1} P((u_w Q) - (u_w C_w)).$$

It follows directly from the definition of peelability that Q is D-peelable if and only if wh Q is defined (that is, the first column of $u_w Q$ contains $u_w C_w$) and wh Q is (wh D)-peelable. Lastly, we wish to define eh Q. For the word a, let $P \searrow (a)$ be the unique column-strict skew tableau (up to translation in the plane) which is Knuth-equivalent to a and whose shape is *antinormal*, that is, the 180° rotation of a partition shape. Let C_e be the last column of D and $u_e \in \sum_r$ the shortest permutation such that $u_e C_e$ is a *final* segment of $[r]$. Define

$$\text{eh } Q = u_e^{-1} P((u_e P \searrow (Q)) - (u_e C_e)).$$

More precisely, this operator slides Q to antinormal shape, acts by the plactic permutation u_e, removes the final segment $u_e C_e$ that is embedded in the last column of $u_e Q$, slides the result to partition shape, and acts by the plactic permutation u_e^{-1}. It will eventually be shown that Q is D-peelable if and only if eh Q is defined and is (eh D)-peelable.

Lemma 54. *Let D be %-avoiding and Q D-peelable. Then* nh Q *is* nh D*-peelable.*

Proof. The proof proceeds by orthodontic induction. Without loss of generality, assume that the first column C of D is nonempty:

Case 1. $D - C <_{ortho} D$. $P(Q - C)$ is $(D - C)$-peelable by definition and nh $P(Q - C)$ is nh$(D - C)$-peelable by induction. Note that nh $P(Q - C) = P(\text{nh}(Q - C)) = P((\text{nh } Q) - (\text{nh } C))$ and $((\text{nh } D) - (\text{nh } C)) = \text{nh}(D - C)$, since Knuth equivalence is preserved under restriction to intervals. Thus, $P((\text{nh } Q) - (\text{nh } C))$ is $((\text{nh } D) - (\text{nh } C))$-peelable. By definition, nh Q is nh D-peelable.

Case 2. $(s_i, id)\, D <_{ortho} D$ for some i.

Case 2a. There is such an i with $i > 1$. In this case s_i and nh commute. Then $s_i Q$ is $(s_i, id)\, D$-peelable by definition and nh $s_i Q$ is nh$(s_i, id)\, D$-peelable by induction. But nh $s_i Q = s_i$ nh Q and nh$(s_i, id)\, D = (s_i, id)$ nh D. Thus s_i nh Q is (s_i, id) nh D-peelable, so by definition nh Q is nh D-peelable.

Case 2b. The only such i is $i = 1$. Let j be minimal such that $(1, j) \in D$. Suppose that $j > 1$. Clearly $(j - 1, j)$ is an inversion of the first *row* of D.

Since D is %-avoiding this means that the $(j-1)$th column of D is properly contained in the jth. By Lemma 51, the shape D can be replaced by $(id, s_{j-1})D$. By induction on j, we may assume that $j=1$. So $(1,1)\in D$. If C is an initial segment, then we are done by Case 1. If not, then C has an inversion of the form $(i', i'+1)$ with $i'>1$, and we are done by Case 2a. ▌

EXAMPLE 55. If nh Q is nh D-peelable it is not necessarily true that Q is D-peelable (same for sh). For example, take

$$D=\begin{array}{c}\times\\ \times\end{array}\qquad Q=1\;\;2\qquad \mathrm{nh}\,D=\begin{array}{c}\cdot\\ \times\end{array}\qquad \mathrm{nh}\,Q=2.$$

LEMMA 56. *Let D be %-avoiding and Q D-peelable. Then* sh Q *is* sh *D-peelable.*

Proof. As usual the proof proceeds by orthodontic induction. Without loss of generality assume that D has no empty columns:

Case 1. $D-C<_{ortho} D$. This case is proven as in Lemma 54.

Case 2. $(s_i, id)\,D<_{ortho} D$ for some i.

Case 2a. There is such an i with $i<r-1$. Then s_i commutes with sh and we are done by induction.

Case 2b. The only such i is $r-1$. If C is an initial segment then Case 1 applies. Otherwise, C must have the form $[1,k]\cup\{r\}$ for some $0\leqslant k<r-1$, or else Case 2a applies. By the %-avoiding property of D, the rth row of D contains each of the $(k+1)$th through $(r-1)$th rows.

Case 2b(i). The rth row of D contains all other rows of D. It follows that the rth row contains a cell in every column of D. By Remark 50 it follows that Q has the letter r in each of its c columns. We wish to show that if $v=s_1s_2\cdots s_{r-1}$ then

$$\mathrm{sh}(Q)=v^{-1}(\mathrm{nh}(vQ)).$$

By direct calculation, it can be shown that the first row of the (v, id) D-peelable tableau vQ consists of c copies of the letter 1, and the rest of the tableau is given by the tableau $1+\mathrm{sh}\,Q$ obtained by adding one to each entry in sh Q. By Lemma 54 $\mathrm{nh}(vQ)=1+\mathrm{sh}\,Q$ is $\mathrm{nh}((v, id)\,D)$-peelable. The shape $\mathrm{nh}((v, id)\,D)$ is obtained by translating the shape sh D one row southward; its first row is empty. Note that $v^{-1}\,\mathrm{nh}(iv, id)\,D)=\mathrm{sh}\,D$; at the time each transposition s_i in v is applied, the ith row is empty. We have $v^{-1}(\mathrm{nh}(vQ))=v^{-1}(1+\mathrm{sh}\,Q)=\mathrm{sh}\,Q$, since in this situation the plastic transpositions s_i in v^{-1} act by replacing each letter $i+1$ by i. Since the shapes sh D and nh$((v, id)D)$ are connected in the orthodontic posed, it follows by the definition of peelability that sh Q is sh D-peelable.

Case 2b(ii). The rth row of D does not contain all others. Let the jth column of D be the leftmost that does not contain a cell in the rth row; this column is nonempty since it is assumed that no column of D is empty. The jth column of D is contained in the first k rows, since the rth row contains each of the rows $(r-1)$ through $k+1$. Let $j' < j$. By applying the %-avoiding property of D to the j'th and jth columns and the rth row and any other row, it follows that the j'th column contains the jth. Thus the jth column is contained in every column to its left. By Lemma 51 the jth column can be repeatedly exchanged with the column to its left until it is the leftmost column; this new shape is $D' = (id, s_1 s_2 \cdots s_{j-1})\, D$, and Q is D'-peelable. But the first column C' of D' is nonempty and contained in the interval $[k] \subseteq [r-2]$. It follows that C' is an initial segment or has an inversion of the form $(i, i+1)$ with $i < r-2$, so we are done by Case 1 or Case 2a. ∎

LEMMA 57. *Let D be a %-avoiding shape and Q D-peelable.*

1. *If the ith row of D contains the $(i+1)$th, then every letter $(i+1)$ in Q is i-paired.*

2. *If the ith row of D is contained in the $(i+1)$th then every letter i in Q is i-paired.*

Proof. The content and number of i-pairs of a word is constant on Knuth equivalence classes. It follows that the number of i-unpaired i's and $(i+1)$'s is as well. By Lemmas 54 and 56, we can remove all rows of D but the ith and $(i+1)$th and replace Q by $P(Q|_{[i,\, i+1]})$. We have reduced to the case where $i = 1$ and $r = 2$, which is easy to check directly. ∎

The treatment of eh is deferred until the proof of the Evac bijection.

The "hat" operations, especially wh and eh, are somewhat difficult to work with since they involve the removal of a set of letters that are supposed to be in a particular column or row of a tableau. We now define operations that are in a sense complementary to the "hat" operations. The operator complementary to wh has already appeared in Proposition 53. Define the operators nb ("north brick"), sb ("south brick"), wb ("west brick"), and eb ("east brick") on %-avoiding shapes and their peelable tableaux as follows. Let D be a %-avoiding shape. Let nb D, sb D, wb D, and eb D be obtained from by filling in the entire first row, last row, first column, and last column of D, respectively. All are %-avoiding. Let $k_n k_s$, k_w, and k_e be the number of cells in the first row, last row, first column, and last column of D, respectively. Let the permutations u_w and u_e be as in the definition of the "hat" operations. Let Q be a D-peelable tableau of partition shape. Define

$$\begin{aligned}\text{nb } Q &= P(Q1^{c-k_n})\\ \text{sb } Q &= P(r^{c-k_s}Q)\\ \text{wb } Q &= u_w^{-1} P([k_w+1, r](u_w Q))\\ \text{eb } Q &= u_e^{-1} P((u_e Q)[r+1-k_e]).\end{aligned}$$

Note that these operations are defined for arbitrary tableaux Q.

Remark 58. By the definition of peelability and Proposition 53, it follows that Q is D-peelable if and only if wb Q is wb D-peelable.

It follows from Lemmas 54 and 56 above and 59 and 60 below, that if Q is D-peelable then nb Q is nb D-peelable and sb Q is sb D-peelable.

PROPOSITION 59. *Let Q be a column-strict tableau of partition shape in the alphabet $[r]$ containing k ones and at most c of any given letter.*

1. *Then $P(Q1^{c-k})$ is the tableau whose first row consists of c copies of the letter* 1 *and whose remainder is given by the tableau $P(Q|_{[2,r]})$. In particular,* nb Q *is obtained by placing a row of c ones atop* nh Q.

2. *If D is a %-avoiding shape then* nb Q *is* (nb D)*-peelable if and only if* nh Q *is* (nh D)*-peelable.*

Proof. Consider the skew column-strict tableau T obtained by placing a row of $c-k$ ones atop Q in the columns $k+1$ through c. The first item follows from considering the jeu-de-taquin $j^X(T)$, where X is the standard tableau of shape (k).

For the second item, consider the cases of the orthodontic-induction, applied to the shape nh D, whose first row is empty:

1. The first column C of nh D is an initial segment in $[2, r]$. So the first column nb C of nb D is an initial segment in $[r]$ and nb$(D-C)=$ nb $D-$nb C. By part 1 the first column of nh Q contains C if and only if the first column of nb Q contains nb C. By applying part 1 to nh Q it follows that nb $P((\text{nh } Q)-C)=P(\text{nb } Q-\text{nb } C)$. By the definition of peelability and induction we are done with this case.

2. C is not initial in $[2, r]$, so that the ith row of nh D is properly contained in the $(i+1)$th for some $i>i$. But nb and s_i commute, so we are done by induction. ∎

PROPOSITION 60. *Let Q be a column-strict tableau tableau of partition shape in the alphabet $[r]$ containing k r's, all lying in the first c columns.*

1. *Then $P(r^{c-k}Q)$ is the tableau obtained by placing a letter r at the bottom of each of the first c columns of $Q|_{[r-1]}$. In particular,* $\mathrm{sb}\, Q$ *is obtained by placing a letter r at the bottom of each of the first c columns of* $\mathrm{sh}\, Q$.

2. *If D is a %-avoiding shape then* $\mathrm{sb}\, Q$ *is* $\mathrm{sb}\, D$*-peelable if and only if* $\mathrm{sh}\, Q$ *is* $\mathrm{sh}\, D$*-peelable.*

Proof. For the first item, it is immediate that the restrictions of Q and $P(r^{c-k}Q)$ to the alphabet $[r-1]$ agree. In Q, all the r's lie in the first c columns. The tableau $P(r^{c-k}Q)$ is given by the column insertion of $c-k$ copies of the letter r into Q. During this process it is obvious that the r's stay in the first c columns.

For the second item, note that the rth row of $\mathrm{sb}\, D$ contains all the others, so that by the definition of peelability, $\mathrm{sb}\, Q$ is $\mathrm{sb}\, D$-peelable if and only if $v\, \mathrm{sb}\, Q$ is $(v, id)(\mathrm{sb}\, D)$-peelable, where $v = s_1 s_2 \cdots s_{r-1}$. Now $(v, id)(\mathrm{sb}\, D) = \mathrm{nb}((v^{-1}, id)\, D)$, and by an argument in Case 2b(i) of Lemma 56, it follows that $v\, \mathrm{sb}\, Q = \mathrm{nb}\, v^{-1} Q$. By the definition of peelability and Lemma 59 we are done. ∎

The following result is the "anti" version of Proposition 53. Let b be a word in the alphabet $[r]$. Define the word $\mathrm{Anti}\, b$ to be that which is obtained from b by replacing each letter i by $r+1-i$ and reversing. This notation is consistent with the notation $\mathrm{Anti}\, Q$ since $\mathrm{Anti}(\mathrm{rowword}(Q)) = \mathrm{rowword}(\mathrm{Anti}(Q))$.

PROPOSITION 61. *Let Q be a column-strict skew tableau of antinormal shape in the alphabet $[r]$ and let $1 \leqslant k \leqslant r$.*

1. *The last column of Q contains $C = [r+1-k, r]$ if and only if the last column of $P \searrow (Q[r-k])$ equals $[r]$.*

2. *If the above holds, then $P \searrow (Q[r-k])$ is the tableau obtained by adjoining a full column $[r]$ to the right of $P \searrow (Q - C)$.*

Proof. Directly from the definitions one can show that the involution Anti preserves Knuth equivalence. The result follows from Proposition 53 by applying the operation Anti. ∎

B.2. *Evacuation Bijection*

THEOREM 62. *Let D be a %-avoiding shape. Then Q is D-peelable if and only if* $\mathrm{Evac}\, Q$ *is* $\mathrm{Evac}\, D$*-peelable.*

Proof. The following are equivalent

1. Q is D-peelable.
2. eh Q is defined and eh Q is eh D-peelable.
3. Evac eh Q is Evac eh D-peelable.
4. wh(Evac Q) is defined and wh Evac Q is wh Evac D-peelable.
5. Evac Q is (Evac D)-peelable.

Consecutive assertions are equivalent by Lemma 64, induction, Lemma 63, and by definition, respectively. ∎

Lemma 63. *Let D be a %-avoiding shape and Q a column-strict tableau of partition shape. Then*

$$\begin{aligned} \text{Evac eb } Q &= \text{wb Evac } Q \\ \text{Evac wb } Q &= \text{eb Evac } Q. \end{aligned} \tag{8.8.1}$$

Also eh Q (*resp.* wh Q) *is defined if and only if* wh (Evac Q) (*resp.* eh(Evac Q) *is, and in this case,*

$$\begin{aligned} \text{Evac eh } Q &= \text{wh Evac } Q \\ \text{Evac wh } Q &= \text{eh Evac } Q. \end{aligned} \tag{8.8.2}$$

Proof. It can be shown directly from the definitions that $s_i \operatorname{Anti} b = \operatorname{Anti} s_{r-i} b$ for any word b in the alphabet $[r]$. It follows that $s_i \operatorname{Evac} Q = \operatorname{Evac} s_{r-i} Q$ for any column-strict tableau Q of partition shape in the alphabet $[r]$. From this (8.8.2) follows. The rest of the result follows from the above observations and Proposition 61. ∎

Lemma 64. *Let D be a %-avoiding shape. Then Q is D-peelable if and only if* eh Q *is defined and* eh Q *is* eh D*-peelable.*

Proof. We use the notation in the definition of eh. By Proposition 61, eh Q is defined if and only if the first column of eb Q equals $[r]$, and in this case, eb Q is obtained from eh Q by adjoining a column $[r]$ on the left. The last column of eb D is maximal, so Lemma 51 applies to show that eh Q is defined and eh D-peelable, if and only if eb Q is eb D-peelable.

The following are equivalent:

1. Q is D-peelable.
2. wh Q is defined and wh D-peelable.
3. eh wh Q is defined and eh wh D-peelable.

4. eb wb Q is eb wb D-peelable.
5. wb eb Q is wb eb D-peelable.
6. wh eh Q is defined and wh eh D-peelable.
7. eh Q is defined and eh D-peelable.

1 and 2 (and 6 and 7) are equivalent by definition. 2 and 3 are equivalent by induction. The equivalences of 3 and 4, and of 5 and 6, follow by the argument above and by Remark 58. The equivalence of 4 and 5 is given by Lemma 65 below. ∎

LEMMA 65. *Let D be a %-avoiding shape and Q a column-strict tableau. Then* eb wb Q *is* eb wb D*-peelable if and only if* wb eb Q *is* wb eb D*-peelable.*

Proof. Note that eb wb $D=$ wb eb D, so that it suffices to show that eb wb $Q=$ wb eb Q, given that either eb wb Q is eb wb D-peelable, or that wb eb Q is wb eb D-peelable. We perform a series of reductions until the shape D is fairly simple, and then verify that eb wb $Q=$ wb eb Q by direct computation.

Again the notation of the definitions of wb and eb is adopted. Without loss of generality we may assume that u_w is the identity and the first column of D is $[k_w]$. Since the plactic permutations in the Young subgroup $\sum_{k_w} \times \sum_{r-k_w}$ of $\sum_r$ commute with wb (see Remark 52) we may assume that $u=u_e$ is a shortest length coset representative. This means that C_e is the union of a final subinterval of $[k_w]$ and a final subinterval of $[k_w+1, r]$, either of which could be empty. Suppose first that C_e is a final segment. Then u is the identity. In this case eb wb Q and wb eb Q are both equal to $P([k_w+1, r]\ Q[r+1-k_e])$ and we are done. Otherwise $C_e=[s, k_w] \cup [t+1, r]$, where $s \leqslant k_w < t \leqslant r$. Then u is the permutation that shuffles the interval $[s, k_w]$ past the interval $[k_w+1, t]$. It certainly suffices to prove the equality of the following two words, since they have P tableaux wb eb Q and eb wb Q, respectively,

$$[k_w+1, r](u^{-1}((uQ)[s-1]))$$
$$u^{-1}((u([k_w+1, r]Q))[s-1]).$$

Since u and u^{-1} do not change any letters outside the interval $[s, t]$, it follows that these two words agree at any position containing a letter outside $[s, t]$. Thus it suffices to show that the restrictions of these two *words* to the interval $[s, t]$ are equal. In light of Lemmas 54 and 56, by removing the rows of D above the sth and the rows below the tth, we may assume that $C_e=C_w$ so that $k_w=k_e=k$, say. Then u is the permutation that shuffles $[k]$ past $[k+1, r]$. It now suffices to show that the words

$$\begin{aligned} a_{we} &= [k+1, r]\, u^{-1}((u \operatorname{columnword}(Q))[r-k]) \\ a_{ew} &= u^{-1}((u[k+1, r] \operatorname{columnword}(Q))[r-k]) \end{aligned} \tag{8.8.3}$$

are equal. This is accomplished by showing that they correspond to the same pair of tableaux under the Robinson–Schensted correspondence. We now show that $P(a_{we}) = P(a_{ew})$, that is, wb eb Q = eb wb Q. Let A be the set of columns that contain a cell of D in a row strictly below the kth. Let $1 \leqslant i \leqslant k < j \leqslant r$. By applying the %-avoiding property of D to the ith and jth rows and to the first column and any other, it follows that the ith row of D contains the jth. Thus, each of the first k rows of D contains each of the last $r-k$. In particular, each of the first k rows contains the set $A' = A \cup \{1, c\}$. Note that for every $j \notin A'$ the jth column is contained in the interval $[k]$ and is therefore contained in the ith column for every $i \in A'$. Now consider the shape eb wb D = wb eb D, which is obtained from D by replacing the first and last columns $[k]$ of D by full columns $[r]$. Starting with eb wb D, there is a sequence of exchanges of nested adjacent columns, that first moves the last column all the way to the left, and then shuffles each of the columns $i \in A$ to the left of each of the columns $j \notin A'$. Call the resulting shape D'. Let Q' be a eb wb D = wb eb D-peelable tableau of partition shape. Equivalently Q' is D'-peelable by Lemma 51. Since the first two columns of D' are equal to $[r]$, by the definition of peelability, the first two columns of Q' are also equal to $[r]$. By Lemma 66, Q' contains key$((m+2)^k)$, and all of the letters in $[k+1, r]$ lie in the rows $[k+1, r]$ of Q'. Q' is then comprised of the following disjoint subtableaux: two columns $[r]$ on the left, a rectangle key$((m^k))$, a northeast subtableau U consisting of numbers in $[k]$, and a southwest tableau T consisting of numbers in $[k+1, r]$. Clearly U has at most k rows and T has at most $(r-k)$ rows and m columns.

Let Q'' (resp. Q) be the column-strict tableau obtained by removing one copy (resp. two copies) of the letter i from the ith row from Q' for $k < i \leqslant r$:

$$Q' = \begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 & \\ 2 & 2 & 2 & \cdots & 2 & U \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ k & k & k & \cdots & k & \\ k+1 & k+1 & & & & \\ k+2 & k+2 & & T & & \\ \vdots & \vdots & & & & \\ r & r & & & & \end{array} \qquad Q = \begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 & \\ 2 & 2 & 2 & \cdots & 2 & U \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ k & k & k & \cdots & k & \\ & & & & & \\ & T & & & & \end{array}$$

By Proposition 67, it follows that

1. Q'' is the unique tableau such that $Q' = \mathrm{wb}\, Q''$.
2. Q'' is the unique tableau such that $Q' = \mathrm{eb}\, Q''$.
3. Q is the unique tableau such that $Q'' = \mathrm{wb}\, Q$.
4. Q is the unique tableau such that $Q'' = \mathrm{eb}\, Q$.

Suppose that $\mathrm{wb}\,\mathrm{eb}\, Q$ is $\mathrm{wb}\,\mathrm{eb}\, D$-peelable. Recall that $\mathrm{wb}\,\mathrm{eb}\, D = \mathrm{eb}\,\mathrm{wb}\, D$. Letting $Q' = \mathrm{wb}\,\mathrm{eb}\, Q$, and arguing as above it follows that $Q' = \mathrm{eb}\,\mathrm{wb}\, Q$, which is clearly $\mathrm{eb}\,\mathrm{wb}\, D$-peelable. The converse is proven the same way. Thus $P(a_{we}) = \mathrm{wb}\,\mathrm{eb}\, Q = \mathrm{eb}\,\mathrm{wb}\, Q = P(a_{ew})$.

To finish the proof it is enough to show that the row-insertion recording tableaux Q_{we} and Q_{ew} of the words a_{we} and a_{ew}, coincide. If D_1 and D_2 are shapes, let $D_1 \otimes D_2$ denote any shape consisting of a translate of D_1 located to the southwest of a translate of D_2 so that these translates lie in disjoint sets of rows and columns. By property Plac1 of plactic permutations, it follows that both of the words a_{we} and a_{ew} are words of column-strict tableaux of shape $(1^{r-k}) \otimes \mathrm{shape}(Q) \otimes (1^{r-k})$. Suppose $\mathrm{shape}(Q)$ has N cells. Let X, Y, and Z denote the successive subintervals of $[N + 2(r-k)]$ of sizes $r-k$, N, and $r-k$, respectively. In light of Proposition 68, it follows that

1. $Q_{we}|_X = Q_{ew}|_X$ is the standard tableau S of shape (1^{r-k}).

2. $Q_{we}|_Y$ and $Q_{ew}|_Y$ are skew standard tableaux in the alphabet Y that are Knuth equivalent to the tableau $\mathrm{Reading}_\lambda$ for the partition $\lambda = \mathrm{shape}(Q)$; see Proposition 69.

3. $Q_{we}|_Z$ and $Q_{ew}|_Z$ are both vertical strips labelled in increasing order from top to bottom by the elements of Z.

It suffices to show that the restrictions of the tableaux Q_{we} and Q_{ew} to the alphabet Z both have shape $\mathrm{shape}(Q')/\mathrm{shape}(Q'')$. For this implies that their restrictions to Y have the same shape. Then $Q_{we}|_Y$ and $Q_{ew}|_Y$ would both be equal to the skew tableau $j_V(K)$, where V is the standard tableau of skew shape $\mathrm{shape}(Q'')/\mathrm{shape}(Q)$ that is increasing from top to bottom.

By the definition of recording tableau and Plac1, we have

$$\mathrm{shape}(Q_{ew}|_{X+Y}) = \mathrm{shape}(\mathrm{wb}\, Q) = \mathrm{shape}(Q'').$$

It follows immediately that

$$\mathrm{shape}(Q_{ew}|_Z) = \mathrm{shape}(Q')/\mathrm{shape}(Q'').$$

By Proposition 68,

$$\text{shape}(j^S(Q_{we}|_{Y+Z})) = \text{shape}(P(Q_{we}|_{Y+Z})) = \text{shape}(\text{eb } Q) = \text{shape}(Q'')$$

and $\text{shape}(P(Q_{we}|_Y)) = \text{shape}(Q)$. This shows that after the jeu-de-taquin that slides the skew tableau $Q_{we}|_{Y+Z}$ to partition shape, the letters in Z form a vertical strip of shape $\text{shape}(Q'')/\text{shape}(Q)$ whose values increase from top to bottom. By reversing this jeu, it follows that $\text{shape}(Q_{we}|_Z) = \text{shape}(Q')/\text{shape}(Q'')$. ▮

LEMMA 66. *Let D be a %-avoiding shape that contains the rectangle $[k] \times [m]$ and has no cells in the rectangle $[k+1, r] \times [m+1, c]$. Let Q be a D-peelable tableau of partition shape. Then Q contains* $\text{key}((m^k))$, *and the letter i appears in the first k rows of Q if and only if $1 \leqslant i \leqslant k$.*

Proof. Let D' be obtained by removing the $(k+1)$th through rth rows of D and let $Q' = Q|_{[k]}$. By Lemma 56 Q' is D'-peelable. The first m columns of D' are initial segments $[k]$. By the definition of peelability, Q' (all of whose letters are less than or equal to k) contains the tableau $\text{key}((m^k))$. It follows that Q also contains this subtableau. The proof now proceeds by orthodontic induction. We may assume that $m > 0$, for otherwise the last $r-k$ rows of D are empty and there is nothing more to prove. If $D - C <_{ortho} D$, then by induction the letters in $[k+1, r]$ lie in rows $[k+1, r]$ in $P(Q-C)$. In the jeu-de-taquin passing from $Q-C$ to $P(Q-C)$, all letters move weakly downward. It follows that the letters in $[k+1, r]$ lie in rows $[k+1, r]$ in $Q-C$. But this is obviously true for the column C in Q as well, so it is true for Q. So we may assume that the first column C of D is not an initial segment. Then C has an inversion of the form $(i, i+1)$, where $i \neq k$ since $m > 0$. If $i < k$ then we are done since i does not move any letters greater than k. If $i > k$ then again we are done since the only letters that change in passing from $s_i Q$ to Q are all strictly greater than k and lie in the rows $[k+1, r]$. ▮

PROPOSITION 67. *Let Q' be a column strict tableau of partition shape in the alphabet $[r]$, that consists of the disjoint union of the following column-strict subtableaux: rectangular key tableau* $\text{key}((m^k))$ *with $1 \leqslant k < r$; on its right, a tableau U of partition shape in the alphabet $[k]$; below, a tableau T of partition shape in the alphabet $[k+1, r]$ with at most m columns; and on the left of all this, a maximal column $[r]$. Let u be the permutation in $\sum_r$ that shuffles the intervals $[k]$ and $[k+1, r]$ past each other. Then*

1. *There is a unique column-strict tableau Q of partition shape such that $Q' = P([k+1, r]Q)$.*

2. *There is a unique column-strict tableau Q of partition shape such that $Q' = P(u^{-1}((uQ)[r-k]))$.*

Furthermore, in both cases the tableau Q is given by removing a letter i from the ith row of Q' for all $k < i \leqslant r$.

Proof. Let $l = r-k$. Let us first prove 1. Any tableau Q satisfying 1 is obtained from Q' by a sequence of reverse column insertions that eject the letter r, then $r-1$, down to $k+1$, in that order. Since the bumping path of a reverse column insertion weakly decreases from right to left, and all of the letters in the first k rows of Q' are in the interval $[k]$, it follows that each of these reverse column insertions must have started in a row i for $i > k$. Since each ejected letter is smaller than the previous one (and there are l of them) it follows that these reverse column insertions had to start at the ends of the rth row, then the $(r-1)$th row, etc., down to the end of the $(k+1)$th row. This means that Q must have the above description, which makes it unique.

For 2, the plactic action of u on Q' can be calculated using the factorization

$$u = (s_l s_{l+1} \cdots s_{r-1}) \cdots (s_2 s_3 \cdots s_{k+1})(s_1 s_2 \cdots s_k). \tag{8.8.4}$$

It follows from the proof of Case 2b(i) in Lemma 56 that uQ' has the following explicit construction. $(uQ')|_{[l]}$ consists of a column $[l]$ adjoined to the left of the tableau $T-k$ given by subtracting k from each entry of T. Then place a column $[l+1, r]$ on the bottom of the first $m+1$ columns of $(uQ')|_{[l]}$. Finally, place the tableau $U+k$ on the right, where $U+k$ is given by adding k to each entry of U.

Next, the word $[l]$ must be row inserted into uQ'. It is easier to see what happens if one first slides uQ' to antinormal shape. $P\searrow(uQ')$ can be computed by the following Knuth equivalences. Consider the column-reading word of uQ'. Let $v^{(j)}$ be the (strictly decreasing) word of the jth column of uQ'. Note that $v^{(1)} = [r]$ commutes with every other column under Knuth equivalence. Since $v^{(2)}$ through $v^{(m+1)}$ contain the subword $[l+1, r]$ and all of the words of the columns of $U+k$ are subwords of $[l+1, r]$, it follows that

$$\begin{aligned}\text{columnword}(uQ') &= [r]\, v^{(2)} v^{(3)} \cdots v^{(m+1)}\, \text{columnword}(U+k)\\ &\sim \text{columnword } U + k v^{(2)} v^{(3)} \cdots v^{(m+2)}[r].\end{aligned}$$

Since the word $v^{(j)}$ for $2 \leqslant j \leqslant m+1$ is given by the concatenation of $[l+1, r]$ with the word of the $(j-1)$th column of $T-k$, it follows that

$$v^{(2)}v^{(3)}\cdots v^{(m+1)} \sim \mathrm{key}((0^l, m^k))\; P\searrow(T-k)$$

$$\mathrm{columnword}(uQ') \sim P\searrow(U+k)\; \mathrm{key}((0^l, m^k))\; P\searrow(T-k)[r]$$

$$P\searrow(uQ') = \begin{array}{ccccc}
 & & & & 1 \\
 & & & & 2 \\
 & & & & \vdots \\
 & & \multicolumn{2}{c}{P\searrow(T-k)} & l-1 \\
 & & & & l \\
 & l+1 & \cdots & l+1 & l+1 \\
 & \vdots & \vdots & \vdots & \vdots \\
P\searrow(U+k) & r-1 & \cdots & r-1 & r-1 \\
 & r & \cdots & r & r
\end{array}$$

By the anti-argument used to prove 1, it follows that the unique column-strict tableau V of antinormal shape such that $P\searrow(V[l]) = P\searrow(uQ')$, is obtained from $P\searrow(uQ')$ by removing the letter i from the ith row for $1 \leqslant i \leqslant l$. V is given by

$$V = \begin{array}{ccccc}
 & & & P\searrow(T-k) & \\
 & l+1 & \cdots & l+1 & l+1 \\
 & \vdots & \vdots & \vdots & \vdots \\
P\searrow(U+k) & r-1 & \cdots & r-1 & r-1 \\
 & r & \cdots & r & r
\end{array}$$

Using Knuth equivalences as above, it follows that $P(V)$ is constructed as follows. $P(V)|_{[l]}$ is equal to the tableau $T-k$. Then place a column $[l+1, r]$ on the bottom of each of the first $m+1$ columns. Then place the tableau $U+k$ on the right. Applying u^{-1} to $P(V)$ using the reverse of the expression (8.8.4), it follows that $u^{-1}P(V) = Q$ where Q is the tableau described above. Therefore Q is the unique column-strict tableau of partition shape such that $Q' = P(u^{-1}((uQ)[l]\downarrow))$. ∎

The following are two well-known facts about recording tableaux.

PROPOSITION 68. *Let a and b be words and A and B alphabets whose cardinalities are given by the lengths of the words a and b, respectively. Consider the recording tableaux $Q(ba)$, $Q(a)$, and $Q(b)$ of the words ba, a, and b, under column insertion, recorded by the alphabets $A+B$, A, and B, respectively. Then*

1. $Q(ba)|_A = Q(a)$.
2. $P(Q(ba)|_B) = Q(b)$.

PROPOSITION 69. *For any partition λ, there is a fixed standard tableau* $\mathrm{Reading}_\lambda$ *depending only on λ such that: A word b satisfies $Q(b) = \mathrm{Reading}_\lambda$ if and only if there is a column-strict tableau T of shape λ such that $b =$* $\mathrm{columnword}(T)$.

B.3. *Box complement Bijection*

THEOREM 70. *Let D be a %-avoiding shape. Then Q is D-peelable if and only if* Boxcomp *Q is* Boxcomp *D-peelable.*

Proof. The proof proceeds by orthodontic induction. It suffices to prove the forward implication, since Boxcomp is involutive on the set of all column-strict tableaux.

Case 1. $D - C <_{ortho} D$. The first column of Q contains the interval C, so by definition the last column of $\mathrm{Boxcomp}\, Q$ contains the interval $[r] - C$, which corresponds to the final interval given by the last column of Boxcomp D. By considering the jeu-de-taquin that slides $\mathrm{Boxcomp}\, Q$ to antinormal shape, it is clear that the last column of $P_\searrow(\mathrm{Boxcomp}\, Q)$ contains $[r] = C$. By Lemma 64 and induction, it is enough to show that Boxcomp wh $Q = $ eb Boxcomp Q, that is, $\mathrm{Boxcomp}(P(Q - C)) = P(\mathrm{Boxcomp}(Q)\, C)$. Note that by taking column-reading words, $Q - C$ and $\mathrm{Boxcomp}(Q)\, C$ can be viewed as sequences of c strictly decreasing words, where the last column of $\mathrm{Boxcomp}(Q)\, C$ is given by the union of the last column of $\mathrm{Boxcomp}(Q)$ and C. As such, the jth column of $Q - C$ is complementary to the $(c + 1 - j)$th column of $\mathrm{Boxcomp}(Q)\, C$. It is shown in [25, Lemma 16] that if $bc \sim b'c'$, where b, c, b', and c' are strictly decreasing words in the alphabet $[r]$, then $([r] - c)([r] - b) \sim ([r] - c')([r] - b')$. It follows that $\mathrm{Boxcomp}(P(Q - C)) = P(\mathrm{Boxcomp}(Q)\, C)$.

Case 2. u_w is not the identity. Observe that u_w is also the shortest permutation that maps the last column of Boxcomp D to a final segment. By induction on the length of u_w and Case 1, it is enough to show that if $(s_i, id)\, D <_{ortho} D$, then $\mathrm{Boxcomp}\, s_i Q = s_i\, \mathrm{Boxcomp}\, Q$. It is easy to see that the tableaux $\mathrm{Boxcomp}\, s_i Q$ and $s_i\, \mathrm{Boxcomp}\, Q$ agree on all letters outside the interval $[i, i + 1]$. Note that the jth column of Q contains an i-unpaired i (resp. $i + 1$) letter if and only if the $(c + 1 - j)$th column of Boxcomp Q contains an i-unpaired $i + 1$ (resp. i). Let $i^k(i + 1)^l$ be the i-unpaired subword of Q. Then $i^l(i + 1)^k$ is the i-unpaired subword of Boxcomp Q. It is now easy to see that $\mathrm{Boxcomp}\, s_i Q = s_i\, \mathrm{Boxcomp}\, Q$. ∎

B.4. *Transpose Bijection*

Let us prove Definition-Proposition 31.

Remark 71. If Q_1 and Q_2 are column-strict tableaux in the alphabet $[r]$ of the same content and the same partition shape, and $\mathrm{nh}\, Q_1 = \mathrm{nh}\, Q_2$, then $Q_1 = Q_2$. For these hypotheses imply that $\mathrm{Evac}\, Q_1 = \mathrm{Evac}\, Q_2$ (see Definition 27)). It follows that the definition of $\mathrm{Tr}_D\, Q$ given in Tr2 is well-defined.

Proof. Without loss of generality it may be assumed that the first column of D is nonempty. Tr_D is shown to be well-defined by orthodontic induction, using the same cases that occur in the proof of Definition-Proposition 13.

Case 1. $D - C <_{ortho} D$ and $(s_i, id)\, D <_{ortho} D$. Let $C = [k]$. By assumption $k \neq i$. Let Q_1 and Q_2 be the definitions of $\mathrm{Tr}_D\, Q$ in Tr2 and Tr3 respectively. Note that since $k \neq i$, wh and s_i commute; see Case 1 of the proof of Definition-Proposition 13. It is easy to check that Q_1 and Q_2 have the same content and same partition shape. By Remark 71, it is enough to show that $\mathrm{nh}\, Q_1 = \mathrm{nh}\, Q_2$. We have

$$\begin{aligned} \mathrm{nh}\, Q_1 &= \mathrm{Tr}_{D-C}(P(Q - C)) = \mathrm{Tr}_{(s_i,\, id)(D-C)}\, s_i P(Q - C) \\ &= \mathrm{Tr}_{((s_i,\, id)\, D) - C}\, P((s_i Q) - C) = \mathrm{nh}\, \mathrm{Tr}_{(s_i,\, id)\, D}(s_i Q) = \mathrm{nh}\, Q_2, \end{aligned}$$

by Tr2 for D, Tr3 for $D - C$, the commutation of wh and s_i, Tr2 for $(s_i, id) D$, and Tr3 for D.

Case 2. $(s_i, id)\, D <_{ortho} D$ and $(s_j, id)\, D <_{ortho} D$ with $i < j$. For the cases below, see the proof of Definition-Proposition 13, which gives the chains of shapes in the orthodontic poset that allow the following computations of peelables.

Case 2a. $j > i + 1$. Since s_i and s_j commute, we have

$$\begin{aligned} s_j\, \mathrm{Tr}_{(s_j,\, id)\, D}\, Q &= s_j s_i\, \mathrm{Tr}_{(s_i s_j,\, id)\, D}\, Q \\ &= s_i s_j\, \mathrm{Tr}_{(s_j s_i,\, id)\, D}\, Q = s_i\, \mathrm{Tr}_{(s_i,\, id)\, D}\, Q, \end{aligned}$$

by various applications of Tr3.

Case 2b. $j = i + 1$. Since the plastic transpositions satisfy the Moore–Coxeter relations, we have

$$\begin{aligned} s_j\, \mathrm{Tr}_{(s_j,\, id)\, D}\, Q &= s_j s_i\, \mathrm{Tr}_{(s_i s_j,\, id)\, D}\, Q \\ &= s_j s_i s_j\, \mathrm{Tr}_{(s_j s_i s_j,\, id)\, D}\, Q = s_i s_j s_i\, \mathrm{Tr}_{(s_i s_j s_i,\, id)\, D}\, Q, \\ &= s_i s_j\, \mathrm{Tr}_{(s_j s_i,\, id)\, D}\, Q = s_i\, \mathrm{Tr}_{(s_i,\, id)\, D}\, Q, \end{aligned}$$

by more applications of Tr3. ∎

It is convenient to add the following covering relations to the orthodontic partial order; the resulting partial order is still denoted $<_{ortho}$:

Or4. If the first row R of D is an initial segment then

$$D - R <_{ortho} D.$$

Or5. If the jth column of D is properly contained in the $(j+1)$th, then

$$(id, s_j)\, D <_{ortho} D.$$

These are merely the transpose relations of Or2 and Or3. Since the family of %-avoiding shapes is stable under Tr, this family could have been characterized by Or1, Or4, and Or5. With the additional relations Or4 and Or5, the poset is no longer graded.

Let Q be D-peelable. Consider the following transpose versions of Tr2 and Tr3:

Tr4. If the first row R of D (that is, the first column of $\mathrm{Tr}\, D$) is an initial segment, then the first column of $\mathrm{Tr}_D\, Q$ contains R (viewed as a single column tableau) and $\mathrm{wh}\, \mathrm{Tr}_D\, Q = \mathrm{Tr}_{\mathrm{nh}\, D}\, \mathrm{nh}\, Q$.

Tr5. If the jth column of D is properly contained in the $(j+1)$th, then $\mathrm{Tr}_D\, Q = s_j\, \mathrm{Tr}_{(id,\, s_j)\, D}\, Q$ (see Lemma 51).

Theorem 72. *The map* Tr_D *is a bijection from the D-peelable tableaux to the* Tr *D-peelable tableaux that is involutive in the sense that* $\mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}\, D = id$. *Furthermore,* Tr_D *satisfies Tr*4 *and Tr*5.

Proof. Let us assume Tr4 and Tr5, and prove by orthodontic induction that for D-peelable tableaux Q, $\mathrm{Tr}_D\, Q$ is $\mathrm{Tr}\, D$-peelable and $\mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}_D\, Q = Q$.

Case 1. The first column of $\mathrm{Tr}\, D$ (that is, the transpose $\mathrm{Tr}\, R$ of the first row R of D viewed as a subshape) is an initial segment. By Tr4, the first column of $\mathrm{Tr}_D\, Q$ contains R (viewed as a single column tableau) and $\mathrm{wh}\, \mathrm{Tr}_D\, Q = \mathrm{Tr}_{\mathrm{nh}\, D}\, \mathrm{nh}\, Q$, which is $\mathrm{Tr}(D - R) = \mathrm{Tr}\, \mathrm{nh}\, D$-peelable by Lemma 54 and induction. Since $\mathrm{Tr}\, \mathrm{nh}\, D = \mathrm{wh}\, \mathrm{Tr}\, D$, we have that $\mathrm{Tr}_D\, Q$ is $\mathrm{Tr}\, D$-peelable by definition

To show that $\mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}_D\, Q = Q$, by Remark 71 it is enough to show that $\mathrm{nh}\, Q = \mathrm{nh}\, \mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}_D\, Q$. We have

$$\begin{aligned} \mathrm{nh}\, \mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}_D\, Q &= \mathrm{Tr}_{(\mathrm{Tr}\, D) - \mathrm{Tr}\, R}\, P((\mathrm{Tr}_D\, Q) - R) \\ &= \mathrm{Tr}_{\mathrm{Tr}(D - R)}\, \mathrm{Tr}_{D - R}\, \mathrm{nh}\, Q = \mathrm{nh}\, Q, \end{aligned}$$

by the definition of $\mathrm{Tr}_{\mathrm{Tr}\, D}$, Tr4, and induction.

Case 2. The jth row (resp. column) of $\mathrm{Tr}\, D$ (resp. D) is properly contained in the $(j+1)$th. By Lemma 51 and induction, $\mathrm{Tr}_{(id,\, s_j)\, D}\, Q$ is $\mathrm{Tr}(id, s_j)\, D = (s_j, id)\, \mathrm{Tr}\, D$-peelable. By Tr3 for the shape $\mathrm{Tr}\, D$ and Tr5, $s_j\, \mathrm{Tr}_{(id,\, s_j) D}\, Q = \mathrm{Tr}_D\, Q$ is $\mathrm{Tr}\, D$-peelable. Moreover,

$$\begin{aligned}\mathrm{Tr}_{\mathrm{Tr}\, D}\, \mathrm{Tr}_D\, Q &= \mathrm{Tr}_{(s_j,\, id)\, \mathrm{Tr}\, D}\, s_j\, \mathrm{Tr}_D\, Q \\ &= \mathrm{Tr}_{\mathrm{Tr}(id,\, s_j)\, D}\, \mathrm{Tr}_{(id,\, s_j)\, D}\, Q = Q,\end{aligned}$$

by the definition of $\mathrm{Tr}_{\mathrm{Tr}\, D}$, Tr5, and induction.

Next Tr5 is proven, assuming that everything is true for orthodontically smaller shapes. Suppose that the jth column of D is properly contained in the $(j+1)$th.

Case 1. $(s_i, id)\, D <_{ortho} D$. We have

$$\begin{aligned}\mathrm{Tr}_D\, Q &= \mathrm{Tr}_{(s_i,\, id)\, D}\, s_i Q \\ &= s_j\, \mathrm{Tr}_{(s_i,\, s_j)\, D}\, s_i Q = s_j\, \mathrm{Tr}_{(id,\, s_j)\, D}\, Q,\end{aligned}$$

by Tr3, Tr5 for the smaller shape $(s_i, id)\, D$ and transposition s_j, and Tr3 for the shape $(id, s_j)\, D$ and the transposition s_i.

Case 2. $D - C <_{ortho} D$.

Case 2a. $j > 1$. In this case C is the first column of $(id, s_j)\, D$ and the actions of nh and s_j on $\mathrm{Tr}_D\, Q$ commute,

$$\begin{aligned}\mathrm{nh}\, \mathrm{Tr}_D\, Q &= \mathrm{Tr}_{D-C}\, P(Q-C) = s_j\, \mathrm{Tr}_{(id,\, s_j)(D-C)}\, P(Q-C) \\ &= s_j\, \mathrm{Tr}_{((id,\, s_j)\, D)-C)}\, P(Q-C) = s_j\, \mathrm{nh}\, \mathrm{Tr}_{(id,\, s_j)\, D}\, Q \\ &= \mathrm{nh}\, s_j\, \mathrm{Tr}_{(id,\, s_j)\, D}\, Q,\end{aligned}$$

by Tr2, Tr5 for the smaller shape $D - C$, the equality of the shapes $((id, s_j)\, D) - C$ and $(id, s_j)(D - C)$, Tr2 for the shape $(id, s_j)\, D$, and the commutation of s_j and nh. Again by Remark 71, Tr5 follows.

Case 2b. $j = 1$. The initial segment C is a proper initial segment in the second column C'.

Case 2b(i). C' is not an initial segment. Then C' has an inversion of the form $(i, i+1)$, with $i > k$ since $C' \supseteq C$. The operation (s_i, id) commutes with removing the first column and with the operation (id, s_1). We have

$$\begin{aligned}\operatorname{Tr}_D Q &= \operatorname{Tr}_{D-C} P(Q-C) = \operatorname{Tr}_{(s_i,\, id)(D-C)} P(s_i(Q-C))\\ &= \operatorname{Tr}_{((s_i,\, id)\, D)-C} P((s_i Q)-C) = \text{nh}\ \operatorname{Tr}_{(s_i,\, id)\, D}\, s_i Q\\ &= \text{nh}\ s_1 \operatorname{Tr}_{(s_i,\, s_1)\, D}\, s_i Q = \text{nh}\ s_1 \operatorname{Tr}_{(id,\, s_1)\, D}\, Q,\end{aligned}$$

by Tr2, Tr3 applied to $D-C$, the commutation of s_i and wh (see Case 1 in the proof of Definition-Proposition 13), Tr2 applied to $(s_i, id)\, D$, induction (this same case with the smaller shape $(s_i, id)\, D$), and Tr3 for the shape $(id, s_j)\, D$. This equality suffices by Remark 71.

Case 2b(ii). C' is an initial segment $[l]$. Since C is a proper subset of C', $l>k$. By Remark 71, it is enough to show that

$$\text{nh}\ \operatorname{Tr}_D Q = \text{nh}\ s_1 \operatorname{Tr}_{(id,\, s_1)\, D}\, Q.$$

By a second application of Remark 71, it is enough to show that

$$\text{shape}(\text{nh}\ \operatorname{Tr}_D Q) = \text{shape}(\text{nh}\ s_1 \operatorname{Tr}_{(id,\, s_1) D}\, Q) \tag{8.8.5}$$

$$\text{nh nh}\ \operatorname{Tr}_D Q = \text{nh nh}\ s_1 \operatorname{Tr}_{(id,\, s_1) D}\, Q, \tag{8.8.6}$$

where nh nh T means $P(T|_{[3,\, c]})$ for a column-strict tableau T of partition shape in the alphabet $[c]$.

Let us prove (8.8.6) first. Consider two chains in the (generalized) orthodontic poset:

$$(id, s_1)\, D - C' - C <_{ortho} (id, s_1)\, D - C' <_{ortho} (id, s_1)\, D <_{ortho} D$$

$$D - C - C' <_{ortho} D - C <_{ortho} D.$$

Both start from the shape $D-C-C' = ((id, s_1)\, D) - C' - C$ and end at D:

$$\begin{aligned}\text{nh nh}\ \operatorname{Tr}_D Q &= \text{nh}\ \operatorname{Tr}_{D-C} P(Q-C) = \operatorname{Tr}_{D-C-C'} P(P(Q-C)-C')\\ &= \operatorname{Tr}_{((id,\, s_1) D)-C'-C} P(P(Q-C')-C)\\ &= \text{nh}\ \operatorname{Tr}_{((id,\, s_1) D)-C'} P(Q-C')\\ &= \text{nh nh}\ \operatorname{Tr}_{(id,\, s_1) D}\, Q = \text{nh nh}\ s_1 \operatorname{Tr}_{(id,\, s_1) D}\, Q,\end{aligned} \tag{8.8.7}$$

by Tr2, Tr2 for the shape $D-C$, the proof of Case 2b(ii) of Lemma 51, Tr2 for the shape $((id, s_1)\, D) - C'$, Tr2 for the shape $(id, s_1)\, D$, and by the fact that the restriction of a tableau to the alphabet $[3, c]$ eliminates the effect of s_1. This proves (8.8.6).

Next (8.8.5) is proven. Define the partitions

$$\begin{aligned} \lambda &= \mathrm{Tr\ shape}(Q) = \mathrm{shape}(\mathrm{Tr}_D\, Q) \\ \mu &= \mathrm{Tr\ shape}(P(Q-(C'+C))) = \mathrm{shape}(\mathrm{nh\ nh\ Tr}_D\, Q) \\ &= \mathrm{shape}(\mathrm{nh\ nh\ Tr}_{(id,\, s_1)\, D}\, Q). \end{aligned}$$

Consider the following chains of partitions:

$$\begin{gathered} \mu \subset \mathrm{shape}(\mathrm{nh\ Tr}_{(id,\, s_1)\, D}\, Q) \subset \lambda \\ \mu \subset \mathrm{shape}(\mathrm{nh}\ s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q) \subset \lambda \\ \mu \subset \mathrm{shape}(\mathrm{nh\ Tr}_D\, Q) \subset \lambda \end{gathered} \tag{8.8.8}$$

These chains can be described in terms of "vacation tableaux," that is, standard tableaux arising from the order of cells vacated during a jeu-de-taquin. The first two chains of partitions in (8.8.8) are given by computing (in stages) the tableaux $\mathrm{nh\ nh\ Tr}_{(id,\, s_1)\, D}\, Q$ and $\mathrm{nh\ nh}\ s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q$, respectively. For this we must determine exactly where the ones and twos occur in $\mathrm{Tr}_{(id,\, s_1)\, D}\, Q$ and $s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q$.

Since $(id, s_1)\, D$ is smaller than D, $\mathrm{Tr}_{(id,\, s_1)\, D}\, Q$ is $\mathrm{Tr}(id, s_1)\, D = (s_1, id)$ $\mathrm{Tr}\, D$-peelable by induction and Lemma 51. By Lemma 57, all of the letters 2 in $\mathrm{Tr}_{(id,\, s_1)\, D}\, Q$ are 1-paired. Since $(\mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[2]}$ has content (l, k), partition shape, and all of its 2's are 1-paired, it must be equal to the tableau $\mathrm{key}((l, k))$. It follows that $(s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[2]} = \mathrm{key}((k, l))$.

Now the intermediate shapes in the first two chains in (8.8.8) can be given. Let $T = \mathrm{stdkey}((l, k))$, $T' = \mathrm{stdkey}((k, l))$, $W = v^T((\mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[3,\, c]})$ $= v^T((s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[3,\, c]})$, and $W' = v^T((\mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[3,\, c]})$. By staring at the jeux, it can be seen that

$$\begin{aligned} \mathrm{shape}(\mathrm{nh\ Tr}_{(id,\, s_1)\, D}\, Q) &= \mu + \mathrm{shape}(W'|_{[k]}) \\ \mathrm{shape}(\mathrm{nh}\ s_1\ \mathrm{Tr}_{(id,\, s_1)\, D}\, Q) &= \mu + \mathrm{shape}(W|_{[l]}). \end{aligned} \tag{8.8.9}$$

Next the relationship between the tableaux W and W' is given. Note that $W = j_{(\mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[3,\, c]}}(T)$ and $W' = j_{(\mathrm{Tr}_{(id,\, s_1)\, D}\, Q)|_{[3,\, c]}}(T')$, so that $P(W) = T$ and $P(W') = T'$. The skew standard tableaux W and W' are *dual Knuth equivalent*, since they are obtained by the same sliding operator from T and T', which are a pair of standard tableaux of the same partition shape and therefore dual Knuth equivalent, so that $Q(W) = Q(W')$ [6, Theorem 2.10].

To obtain information about the shape of $\mathrm{nh\ Tr}_D\, Q$, consider the following chains of shapes:

$$\text{shape}(P(Q-(C'+C))) \subset \text{shape}(P(Q-C)) \subset \text{shape}(Q)$$
$$\text{shape}(P(Q-(C'+C))) \subset \text{shape}(P(Q-C')) \subset \text{shape}(Q).$$

The first and last shapes in both chains are $\mathrm{Tr}\,\mu$ and $\mathrm{Tr}\,\lambda$.

Let $S = \mathrm{tr}\,T$ and $S' = \mathrm{tr}\,T'$ be the ordinary transposes of T and T'. Let $V = v^S(Q-(C'+C))$ and $V' = v^S(Q-(C'+C))$. It is clear that the shapes of V and V' are both equal to $\mathrm{Tr}(\lambda/\mu)$. By staring at the jeux, one can see that

$$\begin{aligned} \text{shape}(P(Q-C)) &= \text{shape}(P(Q-(C'+C))) + \text{shape}(V|_{[l]}) \\ \text{shape}(P(Q-C')) &= \text{shape}(P(Q-(C'+C))) + \text{shape}(V'|_{[k]}). \end{aligned} \tag{8.8.10}$$

Note also that $V = j_{Q-(C'+C)}(S)$ and $V' = j_{Q-(C'+C)}(S')$ so that $P(V) = S$ and $P(V') = S'$. Since S and S' are standard tableaux of the same partition shape, they are *dual Knuth equivalent*. Since V and V' are obtained by applying the same jeu operator $j_{Q-(C'+C)}$ to dual Knuth equivalent tableaux, they are dual Knuth equivalent [6, Theorem 2.10]. It follows that $Q(V) = Q(V')$.

We have $W' = \mathrm{tr}\,V'$, due to (8.8.9) and (8.8.10) and the fact that

$$\mathrm{Tr}_{((id,\,s_1)\,D)-C'}\,P(Q-C') = \mathrm{nh}\,\mathrm{Tr}_{(id,\,s_1)\,D}\,Q.$$

It is enough to show that $\text{columnword}(V) = \text{rev rowword}(W)$, for this implies that

$$\begin{aligned} \text{shape}(\mathrm{nh}\, s_1\, \mathrm{Tr}_{(id,\,s_1)\,D}\,Q) &= \mathrm{Tr}\,\text{shape}(P(Q-C)) \\ &= \text{shape}\,\mathrm{Tr}_{D-C}\,P(Q-C) = \text{shape nh}\,\mathrm{Tr}_D\,Q. \end{aligned}$$

Note that $\text{columnword}(V') = \text{rev rowword}(W')$. We have

$$\begin{aligned} P(\text{rev rowword}(W)) &= \mathrm{tr}\,P(\text{rowword}(W)) = \mathrm{tr}\,T = S = P(\text{columnword}(V)) \\ Q(\text{rev rowword}(W)) &= \mathrm{Evac}\,\mathrm{tr}\,Q(\text{rowword}(W)) \\ &= \mathrm{Evac}\,\mathrm{tr}\,Q(\text{rowword}(W')) = Q(\text{rev rowword}(W')) \\ &= Q(\text{columnword}(V')) = Q(\text{columnword}(V)). \end{aligned}$$

The first equalities follow from the well-known fact that if b is a standard word that corresponds to the pair of standard tableaux (P, Q) under column insertion, then $\mathrm{rev}\,b$ corresponds to $(\mathrm{tr}\,P, \mathrm{Evac}\,\mathrm{tr}\,Q)$ under column insertion. The equalities of the recording tableaux follow from the dual Knuth equivalences of the tableaux V and V', and W and W' [6, Theorem 2.10].

The bijectivity of insertion shows that $\text{rev rowword}(W) = \text{columnword}(V)$. Since it is known that $\text{shape}(W) = \mathrm{Tr}\,\text{shape}(V)$, it follows that $W = \mathrm{tr}\,V$.

This completes the proof of Tr5.

Finally, Tr4 is shown, assuming everything for orthodontically smaller shapes. Suppose the first row R of D is an initial segment.

Case 1. The first column C of D has an inversion $(i, i+1)$. Since the first row of D is initial, $i>1$. In this case s_i and nh commute. By Tr3, $\mathrm{Tr}_D\, Q=\mathrm{Tr}_{(s_i,\, id)\, D}\, s_i Q$. By induction for Tr4 and the shape $(s_i, id)\, D$, the tableau $\mathrm{Tr}_{(s_i,\, id)\, D}\, s_i Q$ is $\mathrm{Tr}(s_i, id)\, D$-peelable and its first column contains R. We have

$$\begin{aligned}\text{wh}\ \mathrm{Tr}_D\, Q &= \text{wh}\ \mathrm{Tr}_{(s_i,\, id)\, D}\, s_i Q\\ &= \mathrm{Tr}_{((s_i,\, id)\, D)-R}\ \text{nh}\ s_i Q=\mathrm{Tr}_{(s_i,\, id)(D-R)}\, s_i\ \text{nh}\ Q\\ &= \mathrm{Tr}_{D-R}\ \text{nh}\ Q\end{aligned}$$

by Tr3, induction for Tr4 and the shape $(s_i, id)\, D$, the commutation of s_i and nh, and Tr3 for $D-R$.

Case 2. $D-C<_{ortho} D$. Let l be the size of R. By Tr2, nh $\mathrm{Tr}_D\, Q= \mathrm{Tr}_{D-C}\, P(Q-C)$ is $\mathrm{Tr}(D-C)=\text{nh Tr}\ D$-peelable. In particular the first column of nh $\mathrm{Tr}_D\, Q$ contains $[2, l]$. Let V be the tableau that gives the reverse order of cells vacated in passing from $\mathrm{Tr}_D\, Q$ to nh $\mathrm{Tr}_D\, Q$ to wh nh $\mathrm{Tr}_D\, Q$. It suffices to show that $\text{shape}(P(V))=(k, 1^{l-1})$, for by Lemma 73 and Tr2, the first column of $\mathrm{Tr}_D\, Q$ contains $[l]$ and nh wh $\mathrm{Tr}_D\, Q$ $=$ wh nh $\mathrm{Tr}_D\, Q=$ wh $\mathrm{Tr}_{D-C}\, P(Q-C)$.

Let W be the skew standard tableau given by the reverse order of cells vacated in passing from Q to wh Q to nh wh Q. Let H be the restriction of Q to the shape $(k, 1^{l-1})$ and T be the standard tableau of shape $(k, 1^{l-1})$ with columnword $T=(k+l-1)(k+l-2)\cdots(l+1)\, 12\cdots l$. By staring carefully at the jeu, one can see that $W=v^T(Q-H)$ or equivalently $W=j_{Q-H}(T)$. So $\text{shape}(P(W))=\text{shape}(T)=(k, 1^{l-1})$.

Note that

$$\begin{aligned}\text{shape}(V) &= \text{shape}(\mathrm{Tr}_D\, Q)/\text{shape}(\text{wh nh}\ \mathrm{Tr}_D\, Q)\\ &= \mathrm{Tr}(\text{shape}(Q)/\text{shape}(\text{nh wh}\ Q))=\mathrm{Tr}\ \text{shape}(W).\end{aligned}$$

Also

$$\begin{aligned}\text{shape}(V|_{[l-1]}) &= \text{shape}(\text{nh}\ \mathrm{Tr}_D\, Q)/\text{shape}(\text{wh nh}\ \mathrm{Tr}_D\, Q)\\ &= \mathrm{Tr}(\text{shape}(\text{wh}\ Q)/\text{shape}(\text{nh wh}Q))=\mathrm{Tr}\ \text{shape}(W|_{[l-1]}).\end{aligned}$$

From this it follows that V and W are ordinary transposes of each other. Therefore,

$$\mathrm{shape}(P(V)) = \mathrm{Tr}\ \mathrm{shape}(P(W)) = \mathrm{Tr}(k, 1^{l-1}) = (l, 1^{k-1}). \quad \blacksquare$$

LEMMA 73. *Suppose Q is a column-strict tableau of partition shape in the alphabet $[r]$ containing k ones.*

1. *Suppose that the first column of Q contains $[l]$. Then the first column of $P(Q|_{[2,r]})$ contains $[2, l]$ and $P(P(Q|_{[2,r]}) - [2, l]) = P(P(Q - [l])|_{[2,r]})$ (this will be denoted* wh nh $Q =$ nh wh Q *by abuse of notation).*

2. *Suppose the first column of* nh Q *contains $[2, l]$. Let V be the skew standard tableau given by the reverse order of cells vacated in passing from Q to* nh Q *to* wh nh Q *(that is, $V|_{[l-1]}$ labels the cells of the vertical strip*

$$\mathrm{shape}(\mathrm{nh}\ Q)/\mathrm{shape}(\mathrm{wh\ nh}\ Q)$$

in increasing order from top to bottom, and $V|_{[l, l+k-1]}$ labels the cells of the horizontal strip shape(Q)/shape(nh Q) *in increasing order from left to right). Then the first column of Q contains $[l]$ if and only if* $\mathrm{shape}(P(V)) = (k, 1^{l-1})$.

Proof. Suppose that the first column of Q contains $[l]$. Let H be the restriction of Q to the shape $(k, 1^{l-1})$. H consists of the column $[l]$ and all of the ones in Q. nh wh $Q = P(Q - H)$ since restriction to the interval $[2, r]$ preserves Knuth equivalence. Now it is shown that wh nh $Q = P(Q - H)$ as well. Let b (resp. c) be the reading words of the part of the first row (resp. column) of the skew tableau $Q - H$. Let A be the subtableau of Q consisting of the second through kth rows with the first letter of each of these rows removed. Let B be the subtableau of Q consisting of all rows strictly south of the lth, again with the first letter in each of these rows removed. Thus the first row of Q is $1^k b$, the second through lth rows have column-reading word $[2, l]$ columnword A, and the remainder has column-reading word c columnword B. Let A' and b' be the first $l-1$ and last rows of the tableau $P(Ab)$, respectively. It is not hard to see that nh Q is given as follows. Its first $l-1$ rows have column-reading word $[2, l]$ columnword A', and the remainder is given by the tableau $P(cBb')$. Thus wh nh $Q = P(cBb'A') = P(cBAb) = P(Q - H)$. This proves 1.

For 2, suppose that the first column of nh Q contains $[2, l]$. Consider the jeu that computes nh Q. Let h denote the "hole" that starts at position $(1, 1)$; it is the only one that changes the first column. Let s be the cell that contains h after h has already exchanged with the letters less than or equal to l but has not yet exchanged with the letters that are strictly greater than l.

If the first column of Q contains $[l]$, it is clear that $s=(l,1)$. If not, then there is some row $j<l$ such that h exchanges straight south to $(j,1)$ and then exchanges eastward into $(j,2)$. In this situation it follows that s is in the first $l-1$ rows. The final position of h is the cell containing the letter l in V.

Next consider the first active hole h' in the jeu that calculates $\mathrm{wh}(\mathrm{nh}\,Q)$, namely, that which starts in the cell $(l-1,1)$ (and replaces the letter l in $\mathrm{nh}\,Q$). Due to the starting position of h', it only exchanges with letters that are greater than or equal to l. Let s' be the position of h' after it has already exchanged with l's but has not yet exchanged with larger letters. It is clear that h' can only exchange eastward with l's (since it originally replaced the letter l), so s' is in the $(l-1)$th row. It follows that s' is in an earlier row than s if and only if Q contains $[l]$. The final position of h' is the cell containing the letter $l-1$ in V.

Let U be the two-letter standard tableau with cells s' and s containing the letters $l-1$ and l, respectively. By performing the above slides in a slightly different order, it can be seen that $V|_{[l-1,l]}=j_{(Q-H)|_{[k+1,r]}}(U)$. Since the jeu is descent-preserving, $l-1$ is a descent of V if and only if it is a descent of U, if and only if s appears in a later row than s', if and only if the first column of Q contains $[l]$. Since V has descents at 1 through $l-2$ and ascents at l through $l+k-2$, it follows that $P(V)$ has these same ascents and descents, and shape$(P(V))$ must be a hook partition $(k,1^{l-1})$ or $(k+1,1^{l-2})$, according as $l-1$ is a descent or ascent of V. ∎

B.5. *Commutation of the Symmetry Bijections*

The commutation relations for the symmetry bijections will follow quite easily from their commutation with the "hat" and "brick" operations:

Evac nh = sh Evac	Boxcomp nh = nb Boxcomp	Tr nh = wh Tr
Evac sh = nh Evac	Boxcomp sh = sb Boxcomp	Tr sh = eh Tr
Evac wh = eh Evac	Boxcomp wh = eb Boxcomp	Tr wh = nh Tr
Evac eh = wh Evac	Boxcomp eh = wb Boxcomp	Tr eh = sh Tr
Evac nb = sb Evac	Boxcomp nb = nh Boxcomp	Tr nb = wb Tr
Evac sb = nb Evac	Boxcomp sb = sh Boxcomp	Tr sb = eb Tr
Evac wb = eb Evac	Boxcomp wb = eh Boxcomp	Tr wb = nb Tr
Evac eb = wb Evac	Boxcomp eb = wh Boxcomp	Tr eb = sb Tr.

Approximately one-fourth of these relations imply all the others, since Evac, Boxcomp, and Tr are involutions and each commutation of a bijection with a "hat" operator is equivalent to its commutation with the corresponding "brick" operator. $\mathrm{Evac}\,\mathrm{nh} = \mathrm{sh}\,\mathrm{Evac}$ holds by the definition of Evac. $\mathrm{Evac}\,\mathrm{wh} = \mathrm{eh}\,\mathrm{Evac}$ follows from Lemma 63. $\mathrm{Boxcomp}\,\mathrm{nh} = \mathrm{nb}\,\mathrm{Boxcomp}$ and $\mathrm{Boxcomp}\,\mathrm{sh} = \mathrm{sb}\,\mathrm{Boxcomp}$ follow more or less by the definition of Boxcomp. $\mathrm{Boxcomp}\,\mathrm{wh} = \mathrm{eb}\,\mathrm{Boxcomp}$ by the proof of Theorem 70. The relation

$$\mathrm{Tr}\,\mathrm{wh} = \mathrm{nh}\,\mathrm{Tr} \tag{8.8.11}$$

follows directly from the definition of Tr. The only other commutation relation needed to prove all the others is $\mathrm{Tr}\,\mathrm{eh} = \mathrm{sh}\,\mathrm{Tr}$. Since both sides produce column-strict tableaux of the same content and same partition shape, it suffices by Remark 71 to show that one obtains equality after applying nh to both sides,

$$\mathrm{nh}\,\mathrm{Tr}\,\mathrm{eh} = \mathrm{Tr}\,\mathrm{wh}\,\mathrm{eh} = \mathrm{Tr}\,\mathrm{eh}\,\mathrm{wh} = \mathrm{sh}\,\mathrm{Tr}\,\mathrm{wh} = \mathrm{sh}\,\mathrm{nh}\,\mathrm{Tr} = \mathrm{nh}\,\mathrm{sh}\,\mathrm{Tr},$$

by the trivial commutation $\mathrm{nh}\,\mathrm{sh} = \mathrm{sh}\,\mathrm{nh}$ (both amount to restriction to the interval $[2, r-1]$ and taking the P tableau since restriction to intervals preserves Knuth equivalence), (8.8.11), induction, Lemma 65, and (8.8.11). This completes the proofs of all the above listed commutation relations between Evac, Boxcomp, Tr, and the "hat" and "brick" operations.

Consider the three commutation relations of Theorem 2. Each is clearly true on the level of shapes. It is also clear that both sides of each of these relations will result in a column-strict tableau of the same content and same partition shape. By Remark 71, it suffices to show that these relations hold when nh is applied to both sides. Also nb works as well, since nb Q is obtained by placing a long row of ones on top of nh Q. First,

$$\begin{aligned}\mathrm{nh}\,\mathrm{Evac}\,\mathrm{Tr} &= \mathrm{Evac}\,\mathrm{sh}\,\mathrm{Tr} = \mathrm{Evac}\,\mathrm{Tr}\,\mathrm{eh}\\ &= \mathrm{Tr}\,\mathrm{Evac}\,\mathrm{eh} = \mathrm{Tr}\,\mathrm{wh}\,\mathrm{Evac} = \mathrm{nh}\,\mathrm{Tr}\,\mathrm{Evac},\end{aligned}$$

where the third equality holds by induction. Second,

$$\begin{aligned}\mathrm{nb}\,\mathrm{Evac}\,\mathrm{Boxcomp} &= \mathrm{Evac}\,\mathrm{sb}\,\mathrm{Boxcomp} = \mathrm{Evac}\,\mathrm{Boxcomp}\,\mathrm{sh}\\ &= \mathrm{Boxcomp}\,\mathrm{Evac}\,\mathrm{sh} = \mathrm{Boxcomp}\,\mathrm{nh}\,\mathrm{Evac}\\ &= \mathrm{nb}\,\mathrm{Boxcomp}\,\mathrm{Evac}.\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{nb}\,\mathrm{Tr}\,\mathrm{Boxcomp} &= \mathrm{Tr}\,\mathrm{wb}\,\mathrm{Boxcomp} = \mathrm{Tr}\,\mathrm{Boxcomp}\,\mathrm{eh}\\ &= \mathrm{Evac}\,\mathrm{Boxcomp}\,\mathrm{Tr}\,\mathrm{eh} = \mathrm{Evac}\,\mathrm{Boxcomp}\,\mathrm{sh}\,\mathrm{Tr}\\ &= \mathrm{Evac}\,\mathrm{sb}\,\mathrm{Boxcomp}\,\mathrm{Tr} = \mathrm{nb}\,\mathrm{Evac}\,\mathrm{Boxcomp}\,\mathrm{Tr}.\end{aligned}$$

APPENDIX C: PROOF OF THEOREM 1

It was shown that Theorem 1 is a consequence of Theorem 20. To establish the latter, it is enough to show that the right-hand side of (3.3.5) or (3.3.3), given by

$$\sum_{\substack{\mathrm{rev}(a) \text{ is } D\text{-peelable} \\ i \text{ is } a\text{-compatible}}} x_i = \sum_{Q \text{ is } D\text{-peelable}} \kappa_{\mathrm{content}(K\_Q)}, \tag{9.9.1}$$

satisfies the conditions S1, S2, and S3 of Theorem 23. Condition S1 is trivial. S3 follows immediately from the following result.

Lemma 74. *Let D be %-avoiding, QD-peelable, and $(s_i, id)\, D \leqslant_{ortho} D$. The contents α and β of the respective tableaux K_Q and K_s_iQ are related by*

$$\beta = \begin{cases} s_i\alpha, & \text{if } \beta_i > \beta_{i+1}, \\ \alpha, & \text{otherwise.} \end{cases}$$

In particular, $\pi_i \kappa_\beta = \kappa_\alpha$.

Proof. It is enough to show that $\alpha_i \leqslant \alpha_{i+1}$ and $\alpha_m = \beta_m$ for $m \notin \{i, i+1\}$. Let us compare the columns of K_Q and K_s_iQ of length j. Let λ be the shape of Q and μ the partition obtained from λ by removing a column of length j. Recall that any column of K_Q of length j, read from bottom to top, is given by the word b ejected during successive reverse column insertions on Q at the ends of the jth, $(j-1)$th, etc., rows. Let T be the resulting tableau of shape μ. Define b' and T' similarly, for s_iQ instead of Q. Clearly $P(bT) = Q$ and $P(b'T') = s_iQ$. Since all of the $(i+1)$'s in Q are i-paired by Lemma 57, the same is true for the Knuth-equivalent word bT. So if i is in b then so is $i+1$. But this holds for all j, so that $\alpha_i \leqslant \alpha_{i+1}$.

Applying s_i, we have

$$P(b'T') = s_iQ = s_iP(uT) = P(s_i(bT)).$$

Now $s_i(bT)$ and $b'T'$ can be viewed as the words of skew tableaux of shape $(1^j) \otimes D(\mu)$. Consider the recording tableaux $Q(bT)$ and $Q(b'T')$. Note that

$$\begin{aligned} \mathrm{shape}(Q(bT)) &= \mathrm{shape}(Q(s_i(bT))) \\ &= \mathrm{shape}(P(s_i(bT))) = \mathrm{shape}(P(b'T')) \\ &= \mathrm{shape}(Q(b'T')). \end{aligned}$$

By Proposition 69, the restrictions of the tableaux $Q(bT)$ and $Q(b'T')$ to the shape $D(\mu)$ are both equal to $\mathrm{Reading}_\mu$. By Pieri's rule, the restrictions of $Q(bT)$ and $Q(b'T')$ to the vertical strip λ/μ are both given by the increasing labelling of λ/μ from top to bottom by the appropriate letters. Therefore, $Q(s_i(bT)) = Q(bT) = Q(b'T')$. By the bijectivity of the Robinson–Schensted correspondence it follows that $s_i(bT) = b'T'$. Since the plastic transposition s_i does not affect letters other than i and $i+1$, it follows that each of the corresponding columns of K_-Q and K_-s_iQ agree, except possibly at cells containing an i or an $i+1$, so that $\alpha_m = \beta_m$ for $m \notin \{i, i+1\}$. ∎

Finally, S2 is an immediate consequence of the following lemma.

Lemma 75. *Let D be a %-avoiding shape whose first column is the initial segment $C = [k]$. Consider the map*

$$\begin{array}{c} \{(a, i)\colon i \text{ is } a\text{-compatible and } a \text{ contains the subword } 12\cdots k\} \\ \phi\Big\downarrow \\ \{(\hat{a}, \hat{\imath})\colon \hat{\imath} \text{ is } \hat{a}\text{-compatible}\} \end{array} \tag{9.9.2}$$

given by $\phi(a, i) = (\hat{a}, \hat{\imath})$, where $\hat{a}$ (resp. $\hat{\imath}$) is obtained by removing the rightmost occurrence of p from a (resp. i) for all $1 \leqslant p \leqslant k$. Then ϕ is a bijection that restricts to a bijection ϕ_D

$$\begin{array}{c} \{(a, i)\colon i \text{ is } a\text{-compatible and } \mathrm{rev}(a) \text{ is } D\text{-peelable}\} \\ \phi_D\Big\downarrow \\ \{(\hat{a}, \hat{\imath})\colon \hat{\imath} \text{ is } \hat{a}-\text{compatible and } \mathrm{rev}(\hat{a}) \text{ is } (D-C)\text{-peelable}\}. \end{array} \tag{9.9.3}$$

Proof. It is easy to verify from the definitions that if i is a-compatible and a contains the subword $12\cdots k$, then i must contain these letters in the same positions, and $\phi(a, i) = (\hat{a}, \hat{\imath})$ has the property that $\hat{\imath}$ is $\hat{a}$-compatible. It is also straightforward to check that the map ϕ has an inverse: given $(\hat{a}, \hat{\imath})$ such that $\hat{\imath}$ is $\hat{a}$-compatible, let a (resp. i) be the word obtained from $\hat{a}$) (resp. $\hat{\imath}$) by adding the letter p in the position to the right of all letters less than or equal to p and to the left of all letters greater than p, for all $1 \leqslant p \leqslant k$.

If $\mathrm{rev}(a)$ is D-peelable, then $P(\mathrm{rev}(a))$ is D-peelable and its first column contains $[k]$. By Proposition 53, $\mathrm{rev}(a)$ contains the subword $[k]$ and so a contains the subword $12\cdots k$.

It remains to show that $\mathrm{rev}(a)$ is D-peelable if and only if $\mathrm{rev}(\hat{a})$ is $(D-C)$-peelable. Let Q be a skew tableau whose shape is a horizontal strip, whose pth row is the weakly increasing word consisting of the entries of a whose corresponding values in i equal p. Clearly $\mathrm{rowword}(Q)=\mathrm{rev}(a)$. By the definition of compatibility, each of the entries in the pth row of Q is at least p, for all p. Furthermore, the leftmost entry of the pth row of Q is p, for all $1 \leqslant p \leqslant k$. Let Q' be the tableau obtained from Q by replacing the leftmost entry (which is p) in the pth row of Q by the letter p', for $1 \leqslant p \leqslant k$, where the primed numbers are declared to be smaller than the plain numbers, and let $\mathrm{rowword}(Q')=\mathrm{rev}(a')$. When $P(Q)$ is computed by the jeu de taquin, the leftmost entries in the first k rows slide straight across to form the topmost entries C in the first column of $P(Q)$. Consequently, it is not hard to see that when $P(Q')$ is computed by the jeu de taquin with respect to the enlarged alphabet, the same slides are performed, with the leftmost entries in the first k rows sliding straight across to form the column C'. That is, $P(Q)$ and $P(Q')$ agree except in the first k letters of the first column, where $P(Q)$ contains C and $P(Q')$ contains C'. We have

$$\begin{aligned} P(P(Q)-C) &= P(P(Q')-C') = P(P(Q')|_{plain}) \\ &= P(P(\mathrm{rev}(a'))|_{plain}) = P(\mathrm{rev}(a')|_{plain}) = P(\mathrm{rev}(\hat{a})), \end{aligned}$$

where $|_{plain}$ denotes the operation of erasing the primed letters. The first equality has already been established and the second is obvious. The third equality holds by abuse of notation since $\mathrm{rowword}(Q')=\mathrm{rev}(a')$. The fourth equality follows from the fact that Knuth equivalence is preserved under restriction to interval subalphabets. The final equality holds since $\mathrm{rev}(a')|_{plain} = \mathrm{rev}(\hat{a})$. Since $P(P(Q)-C)$ and $P(\mathrm{rev}(\hat{a}))$ are Knuth equivalent tableaux of partition shape, they must be equal. But $\mathrm{rowword}(Q)=\mathrm{rev}(a)$, so that $\mathrm{rev}(a)$ is D-peelable if and only if $\mathrm{rev}(\hat{a})$ is $(D-C)$-peelable. ∎

EXAMPLE 76. Here is an example illustrating the proof of Lemma 75 with $k=2$.

$$D = \begin{array}{ccccc} \times & & & & \\ \times & \times & & & \\ & \times & \times & \times & \\ & \times & \times & & \times \\ & & \times & & \end{array}$$

$$\begin{pmatrix} a \\ i \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 & 3 & 2 & 4 & 4 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \hat{a} \\ \hat{\imath} \end{pmatrix} = \begin{pmatrix} 3 & 1 & \cdot & 3 & \cdot & 4 & 4 & 3 & 5 \\ 1 & 1 & \cdot & 2 & \cdot & 3 & 3 & 3 & 4 \end{pmatrix}$$

$$Q = \begin{array}{ccccccccc} & & & & & & 1 & 1 & 3 \\ & & & & 2 & 3 & & & \\ & 3 & 4 & 4 & & & & & \\ 5 & & & & & & & & \end{array} \qquad Q' = \begin{array}{ccccccccc} & & & & & & 1' & 1 & 3 \\ & & & & 2' & 3 & & & \\ & 3 & 4 & 4 & & & & & \\ 5 & & & & & & & & \end{array}$$

$$P(Q) = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & \\ 5 & & \end{array} \qquad P(Q') = \begin{array}{ccc} 1' & 1 & 3 \\ 2' & 3 & 4 \\ 3 & 4 & \\ 5 & & \end{array}$$

C.1. *Proof of Theorem* 44

It will be shown that $\mathscr{B}_D^{\text{flag}}$ satisfies the defining recurrence of $\mathscr{M}_D$. This is accomplished by studying the bijections that take each of the following sets to the next.

1. $T \in \mathscr{B}_D^{\text{flag}}$.
2. (P, Q_*) such that Q_* is Tr D-peelable and $K_+ P \leqslant K_-(\mathrm{Tr}_{\mathrm{Tr}\, D}\, Q_*)$.
3. (P, Q) such that Q is D-peelable and $K_+ P \leqslant K_- Q$.
4. (b, j) such that j is weakly decreasing, $b_p \leqslant b_{p+1}$ whenever $j_p = j_{p+1}$, $b_p \leqslant j_p$ for all p, and $\mathrm{std}^{-1}(Q(b), \mathrm{content}(j))$ is D-peelable.
5. (a, i) such that $\mathrm{rev}(a)$ is D-peelable, and i is a-compatible.

The bijections are given by:

$1 \to 2$: $P = P(\mathrm{fillingword}(T))$, $Q_* = Q_*(T)$. This is bijective by definition.

$2 \to 3$: $Q = \mathrm{Tr}_{\mathrm{Tr}\, D}\, Q_*$. This is bijective by Theorem 72.

$3 \to 4$: j is the weakly decreasing word with $\mathrm{content}(j) = \mathrm{content}(Q)$, $P(b) = P$, and $Q(b) = \mathrm{std}(Q)$. This is a bijection by Lemma 77 below.

$4 \to 5$: The pair of words (a, i) is obtained by viewing the pair of words (b, j) as a sequence of biletters (b_p, j_p) and sorting the flipped biletters (j_p, b_p) into weakly increasing order from left to right, where $(x, y) < (x', y)$ if $x < x'$, or $x = x'$ and $y > y'$. This is easily seen to be bijective. Note that the biwords (b, j) and $(\mathrm{rev}(a), \mathrm{rev}(i))$ are generalized inverses, so that $P(b) = \mathrm{std}^{-1}(Q(\mathrm{rev}(a)), \mathrm{content}(i))$ and $P(\mathrm{rev}(a)) = \mathrm{std}^{-1}(Q(b), \mathrm{content}(j))$.

Lemma 77. *Let Q be a column-strict tableau of partition shape and j the weakly decreasing sequence with* $\mathrm{content}(j) = \mathrm{content}(Q)$. *Let $\mathscr{W}_Q$ be the set of words b of the same length as j such that $b_p \leqslant j_p$ for all p, and* $Q(b) = \mathrm{std}(Q)$.

Then the map $b \mapsto P(b)$ is a bijection from $\mathscr{W}_Q$ to the set of tableaux P such that $K_+ P \leqslant K_- Q$.

Proof. The above map from 4 to 5 gives a bijection from $\mathscr{W}_Q$ to the pairs (a, i) such that $P(\mathrm{rev}(a)) = Q$ and i is a-compatible. By [23, Lemmas 8 and 9], one can replace Q by the key tableau $K_- Q$ without loss of generality. The result follows by induction on the content of this key tableau, using the ideas of the proof of [23, Theorem 2]. ∎

The proof of Theorem 44 is now given.

Proof. It is shown that $\mathscr{B}_D^{\mathrm{flag}}$ satisfies the defining recurrence for $\mathscr{M}_D$. For $D = \varnothing$ the result is trivial. Suppose that $D - C <_{ortho} D$ with $C = [k]$. There is a bijection $T \mapsto (a, i) \mapsto (\hat{a}, \hat{\imath}) \mapsto \hat{T}$. where the first and third maps are the above composite map 1 to 5 for D and its inverse for $D - C$, and the middle map is given by Lemma 75. It must be shown that fillingword$(T) =$ $[k]$ fillingword$(\hat{T})$. Consider the biwords (b, j) and $(\hat{b}, \hat{\jmath})$ that occur as intermediate results in the above bijection. Let $b^{(p)}$ (resp. $\hat{b}^{(p)}$) be the subword of b (resp. $\hat{b}$) that occupies the same positions as the letters p in j (resp. $\hat{\jmath}$); it consists of letters that are less than or equal to p by definition. Then $b^{(p)} = \hat{b}^{(p)}$ for $p > k$ and $b^{(p)} = \hat{b}^{(p)} p$ for $1 \leqslant p \leqslant k$. By direct computation it can be shown that

$$\begin{aligned}
\text{fillingword}(T) \sim b &= \cdots \hat{b}^{(k+2)} \hat{b}^{(k+1)} (\hat{b}^{(k)} k) \cdots \hat{b}^{(2)} 2 \hat{b}^{(1)} 1 \\
&\sim \cdots \hat{b}^{(k+2)} \hat{b}^{(k+1)} \hat{b}^{(k)} \cdots \hat{b}^{(1)} k (k-1) \cdots 21 \\
&= \hat{b}[k] \sim \text{fillingword}(\hat{T})[k].
\end{aligned}$$

Let $Q'_* = \mathrm{std}^{-1}(\mathrm{tr}\, Q(\text{fillingword}(\hat{T})[k]), \beta)$, where β_p is the number of cell in the pth column of D. It is enough to show that $Q'_* = Q_*$. Now

$$\mathrm{nh}\, Q'_* = \hat{Q}_* = \mathrm{Tr}_{\mathrm{wh}\, D}\, \hat{Q} = \mathrm{Tr}_{\mathrm{wh}\, D}\, \mathrm{wh}\, Q = \mathrm{nh}\, \mathrm{Tr}_D\, Q = \mathrm{nh}\, Q_*,$$

by Lemma 68, definition, the proof of Lemma 75, Tr2, and definition. Remark 71 shows that $Q'_* = Q_*$, so that fillingword$(T) =$ fillingword$(\hat{T})[k]$.

Suppose that $(s_i, id)\, D <_{ortho} D$. Let $T \in \mathscr{B}^{\mathrm{flag}}_{(s_i,\, id) D}$. Suppose f_i^p(fillingword(T)) is defined. Let T' be the filling of D such that fillingword$(T') =$ f_i^p(fillingword(T)). Denote all images under the bijections from set 1 through 5 of T' by priming the corresponding images of T. Since f_i does not affect the recording tableau, $Q'_* = Q_*$. Also $P(f_i^p(\text{fillingword}(T))) =$ $f_i^p(P(\text{fillingword}(T)))$, so that $P' = f_i^p(P)$. By Tr3, $s_i Q' = Q$. Let $(\tau_i b, j'\mathrm{E})$ be the pair of words in set 4 for the shape D, corresponding to the tableau pair $(P, s_i Q = Q')$ in set 3. By Lemma 77, it suffices to show that for every (s_i, id) D-peelable tableau Q,

$$\mathscr{W}_{s_i Q} = \bigcup_{b \in \mathscr{W}_Q} \pi_i(\tau_i b).$$

By the proof of Lemma 77, without loss of generality one may reduce to the case where Q is a key tableau. This case is settled in the proof of [23, Theorem 2]. ∎

APPENDIX D: PROOF OF THEOREM 3

It suffices to establish the bijection g of Theorem 39. Northwest shapes allow us to make some simplifying assumptions. Let D be a northwest shape. If the first column C of D has the inversion $(i, i+1)$, then the ith row of D is empty. In this case, if Q is D-peelable, the map $Q \mapsto s_i Q$ is given by the trivial relabelling that merely replaces each letter $i+1$ by i. Therefore we may assume without loss of generality that every row and column of D is nonempty and C is an initial segment, say $[k]$. These assumptions will be made in the following proofs.

The following lemma is the first step in showing that the map g is well-defined when $|X| = 1$.

LEMMA 78. *Let D be northwest with rows in initial segment order, Q a D-peelable tableau of partition shape, and y a corner cell of* shape(Q). *Let i be the letter ejected by the reverse column insertion on Q at y. Then there is a corner cell in the ith row of D.*

Proof. If there is no corner cell in the ith row of D, it follows from the northwest property of D and the initial segment order on the rows of D, that the ith row is contained in the next nonempty row, which we may assume without loss of generality is the $(i+1)$th. By Lemma 57, every i in Q is i-paired. Then a Knuth equivalence of the form $Q \sim i\hat{Q}$ is impossible, for the first letter i of $i\hat{Q}$ is i-unpaired, and the number of i-unpaired i's is constant on Knuth equivalence classes. ∎

It is convenient to introduce an alternate definition of peelability for northwest shapes. Let D be a northwest shape, $(k, 1)$ the cell at the bottom of its first column, and D^h the northwest shape $D - \{(k, 1)\}$. Given a tableau Q, say that Q^h is defined if k appears in the first column of Q. In this case let Q^h be the tableau obtained by replacing this occurrence of k by a jeu-de-taquin hole and sliding this hole to the southeast border, i.e.

$Q^h = P(Q-k)$, where $Q-k$ means the row- or column-reading word of the punctured tableau $Q-k$.

LEMMA 79. *Let D be a northwest shape. A tableau Q of partition shape is D-peelable if and only if Q^h is defined and is D^h-peelable.*

Proof. Suppose that Q is D-peelable. Then its first column contains $[k]$ and in particular k, so Q^h is defined. Let $C^h = C-k$. Q^h is D^h-peelable, since $p(Q^h - C^h) = P(Q-C)$ and $D^h - C^h = D - C$.

Conversely, suppose Q^h is defined and D^h-peelable. Then the first column of Q^h contains $C^h = [k-1]$. Since the jeu calculating Q^h only moves letters that are at least as large as k, we have $Q^h|_{[k-1]} = Q|_{[k-1}$, so that the first column of Q contains $[k-1]$ as well. Since Q^h is defined, the first column of Q also contains k. Let X be the standard tableau of shape (1^k) and x the restriction of X to the cell $(k, 1)$. It is easily seen that

$$P(Q-C) = j^X(Q-C) = j^{X-x}(j^x(Q-C)) = j^{X-x}(Q^h - C^h) = P(Q^h - C^h).$$

Then Q is D-peelable since $P(Q^h - C^h) = P(Q-C)$ is wh $D^h =$ wh D-peelable. ∎

The following result, together with Lemma 78, shows that g is a bijection when $|X| = 1$. Say that a cell of a northwest shape D is a *pseudo-corner cell* if it is at the bottom of its column and is leftmost in its row with this property. If the rows of D are in initial segment order this is the same thing as a corner cell.

LEMMA 80. *Let D be a northwest shape whose ith row contains a pseudo-corner cell y. Suppose Q and Q^- are two tableaux of partition shape related by $Q \sim iQ^-$. Then Q is D-peelable if and only if Q^- is D^--peelable, where $D^- = D - y$.*

Proof. Let T and T^- be the tableaux given by removing the first columns of Q and Q^-, respectively. Note that Q is given by the column insertion of the letter i into Q^-. Let z be the letter in the first column of Q^- that is bumped by i (if i does not bump any letter, let z be the empty word). Then by the definition of column insertion, $T = P(zT^-)$. The first column of Q and Q^- differ at precisely one cell (say $(i', 1)$) which contains the letter i in Q and contains the letter $z \geqslant i$ in Q^-. The proof divides into several cases.

Case 1. $i = k$. Then $y = (k, 1)$ and wh $D =$ wh D^-. The first column of Q contains $[k]$ if and only if the first column of Q^- contains $[k-1]$. Without loss of generality we may assume this holds. We may write columnword(Q) $= b[k]\ T$ and columnword(Q^-) $= bz[k-1]\ T^-$, where b is a strictly decreasing word. Then

$$\text{wh } Q^- = P(Q^- - [k-1]) = P(bzT^-) = P(bT) = P(Q - [k]) = \text{wh } Q,$$

which suffices by the definition of peelability.

Case 2. $i > k$. Here the first columns of D and D^- agree, and the ith row of wh D has the pseudo-corner cell y. So $(\text{wh } D)^- = (\text{wh } D) - y = \text{wh } D^-$.

The first column of Q contains $[k]$ if and only if the first column of Q^- does. Again we may suppose this holds. Then we may write columnword(Q) $= cib[k]\ T$ and columnword(Q^-) $= czb[k]\ T^-$. Using Knuth equivalences and column-reading words, we have

$$i \text{ wh } Q^- = i(Q^-[k]) = iczbT^- \sim cibzT^- \sim cibT = Q - [k] \sim \text{wh } Q,$$

that is, $(\text{wh } Q)^- = \text{wh } Q^-$. It follows that wh Q is defined and is wh D-peelable, if and only if $(\text{wh } Q)^-$ is defined and is $(\text{wh } D)^-$-peelable, if and only if wh Q^- is defined and is wh D^--peelable, by induction and the commutation of $^-$ and wh on Q and D.

Case 3. $i < k-1$. Again y is a pseudo-corner cell of D^h, so that $D^{h-} = D^{-h}$. By Lemma 79 and induction, it suffices to show that $Q^{-h} = Q^{h-}$, that is, $P(iQh) = Q^{-h}$. Recall that $Q(i', 1) = i$. Only the first i' rows of Q are changed in passing to Q^-, since the bumping path of a column insertion proceeds weakly north. This shows that the first column of Q contains k if and only if the first column of Q^- does. Again we may assume this holds. The same cell in the first columns of Q and Q^- contain the letter k; call this cell $(i'', 1)$ with $i'' > i'$ since $k > i$. The jeu that computes Q^h and Q^{-h} only affects the rows weakly south of the i''th, since the holes slide to the southeast. It follows that $Q^{-h} = Q^{h-}$.

Case 4. $i = k-1$. Since the ith row of D contains the pseudo-corner cell y, there is a nonempty column of D that is not equal to $[k]$; let the $(c+1)$th be the leftmost such column. Let $K = \text{key}((c^k))$. By abuse of notation let K also represent the subshape of D given by its first c columns, each of which is given by $[k]$.

If the $(c+1)$th column of D is an initial segment in the first column $[k]$ of D or vice versa, let $D' = (id, (1c+1))\ D$ be the shape obtained by exchanging the first and $(c+1)$th columns of D. Note that y is a pseudo-corner cell of D' and $D'^- = D' - y = (id, (1(c+1)))(D^-)$. Then Q is D-peelable, if and only if Q is D'-peelable, if and only if Q^- is D'^--peelable, if and only if Q^- is D^--peelable, by Lemma 51, one of the previous cases, and Lemma 51.

So assume that the $(c+1)$th column is not an initial segment of $[k]$ or vice versa. Let s be minimal such that $(s, c+1) \notin D$. $s \leqslant k$, for otherwise the

$(c+1)$th column would contain the first column $[k]$ as an initial segment. Each of the first k rows of D (in particular the sth) contains the interval $[c]$. By the northwest property, for every inversion (p, q) of the $(c+1)$th column of D, the pth row is contained in the interval $[c]$. In particular this is true of the sth row, which must then be equal to $[c]$. The sth row is a proper initial segment in each of the rows above it. The sth row is also weakly above and contained in the kth, so that it cannot contain a pseudo-corner cell. Thus $s \neq i$.

Case 4a. $s=1$. It is enough to show that the following are equivalent:

(a) Q is D-peelable.

(b) Q contains K and $P(Q-K)$ is $(D-K)$-peelable.

(c) Q contains K and $P(\mathrm{nh}\, Q - \mathrm{nh}\, K)$ is $(\mathrm{nh}\, D - \mathrm{nh}\, K)$-peelable.

(d) Q contains K and $\mathrm{nh}\, Q$ is $\mathrm{nh}\, D$-peelable.

(e) Q^- contains K and $\mathrm{nh}\, Q^-$ is $\mathrm{nh}\, D^-$-peelable.

(f) Q^- contains K and $\mathrm{nh}\, Q^-$ is $\mathrm{nh}\, D^-$-peelable.

(g) Q^- contains K and $P(Q^- - K)$ is $(D^- - K)$-peelable.

(h) Q^- is D^--peelable.

The equivalences of items (a) and (b); (c) and (d); (g) and (h), all follow from the definition of peelability where wh is applied c times.

(b) $\Leftrightarrow$ (c) Note that $D-K = \mathrm{nh}\, D - \mathrm{nh}\, K$. It is enough to show that $P(\mathrm{nh}\, Q - \mathrm{nh}\, K = P(Q-K)$, assuming that Q contains K. This follows from repeated applications of Lemma 54. (f) $\Leftrightarrow$ (g) follows similarly.

(e) $\Leftrightarrow$ (f) Since $i \neq s = 1$ and the bumping path of the column insertion of i into Q^- consists of letters that are weakly greater than i, it follows that

$$i(\mathrm{nh}\, Q^-) \sim i(Q^-|_{[2,r]}) = (iQ^-)|_{[2,r]} \sim Q|_{[2,r]} \sim \mathrm{nh}\, Q,$$

that is, $\mathrm{nh}\, Q^-$ makes sense and equals $\mathrm{nh}\, Q^-$.

(d) $\Leftrightarrow$ (e) The previous computation shows that $(\mathrm{nh}\, Q)^-$ makes sense. By induction applied to $\mathrm{nh}\, D$, $\mathrm{nh}\, Q$ is $\mathrm{nh}\, D$-peelable if and only if $(\mathrm{nh}\, Q)^-$ is $(\mathrm{nh}\, D)^-$-peelable. It remains to show that Q contains K if and only if Q^- does, but this follows easily from the fact that $k-1 = i < k$.

Case 4b. $1 < s < i$. Recall that the sth row is contained in every row above it. It is clear that exchanging the sth and $(s-1)$th rows of D commutes with removing the cell y. Moreover, $s_{s-1} iQ^- = is_{s-1}Q^-$, that is, $(s_{s-1}Q)^-$ is defined and equals $s_{s-1}(Q^-)$. This suffices by the definition of peelability and induction.

Case 4c. $s > k$. This has already been ruled out.

Case 4d. $s=k$. Recall that the sth row equals $[c]$. Let $D\backslash k$ be the shape obtained from D by removing all cells in the kth row. Let $Q\backslash k$ be the tableau obtained by the removal of the letters k from Q, creating a tableau with holes, and a jeu de taquin that slides the holes (in order from right to left) to the southeast boundary. Suppose that Q is D-peelable. The ith row of D contains cells in the first $c+1$ columns and at least one additional cell, namely, y. The proof proceeds by proving the equivalence of the eight assertions in Case 4a, except that nh is replaced by $\backslash k$. The only step that requires a different argument is $5 \Leftrightarrow 6$, namely, that $(Q\backslash k)^-$ is $(D\backslash k)^-$-peelable if and only if $Q^-\backslash k$ is $D^-\backslash k$-peelable, assuming that Q^- contains K. Since $i<k$, it is easy see that Q contains K as well, and that all of the k's in Q and Q^- are contained in the kth row of the subtableau K. The jeu de taquin that calculates $Q\backslash k$ from Q only changes the rows weakly south of the kth since the holes all start in the kth row and slide to the southeast. The bumping path of the column insertion of i into Q^- starts at the cell $(i, 1)$ and proceeds weakly north. Therefore these operations commute, so that $(Q\backslash k)^-$ is defined and $i(Q^-\backslash k) \sim (iQ^-)\backslash k = Q\backslash k$, that is, $(Q\backslash k)^- = Q^-\backslash k$. ∎

We now show that g is a well-defined bijection. A *pseudo-corner cell* of a northwest shape is one that is at the bottom of its column and is leftmost in its row with respect to this property.

Let D be northwest with rows in initial segment order, QD-peelable, X a vertical strip of cardinality m in shape(Q), with $i_1 i_2 \cdots i_m$ and Q^- as in the definition of g. In light of Lemma 80 the following two lemmas suffice.

LEMMA 81. *Let D be a northwest shape with rows in initial segment order, $C=\{i_1, ..., i_m\}$ a set of row indices in decreasing order. Then C is the row indices of some vertical strip $Y=\{y_1, ..., y_m\}$ in D if and only if for each t, the i_{t+1}th row of the shape $D\{y_1, ..., y_t\}$ contains a pseudo-corner cell, namely, y_{t+1}.*

Proof. The proof follows directly from the definition of a vertical strip by induction on t. ∎

LEMMA 82. *Let D be a northwest shape with rows in initial segment order, Q D-peelable, and $Q=P(i_1 i_2 \cdots i_m Q^-)$ for $i_1 > i_2 > \cdots > i_m$. Then $C=\{i_1, i_2, ..., i_m\}$ is the set of row indices of a vertical strip of D.*

Proof. Lemma 78 suffices when $m=1$. Suppose the lemma does not hold. By Lemma 81, without loss of generality D has a vertical strip $Y=\{y_1, ..., y_{m-1}\}$ with row indices $i_1 > \cdots > i_{m-1}$, but the i_mth row of $D-Y$ does not contain a pseudo-corner cell, for some $m \geqslant 2$.

Case 1. $i_{m-1} > i_m + 1$. In this case the i_mth and $(i_m + 1)$th rows of $D - Y$ are the same as those of D. Since the i_mth row of $D - Y$ does not contain a pseudo-corner cell, it must be contained in the $(i_m + 1)$th row of $D - Y$. But this would imply that the i_mth row is contained in the $(i_m + 1)$th in D. By Lemma 57, all of the i_m's in Q are i_m-paired. This contradicts $Q = P(i_1 i_2 \cdots i_m Q^-)$.

Case 2. $i_{m-1} = i_m + 1$. Let $y_{m-1} = (i_{m-1}, j)$. There cannot be a cell $(i_m, j) \in D - Y$ just above y_{m-1}, for otherwise y_{m-1} would be at the bottom of its column in $D - Y$ and the i_mth row of $D - Y$ would contain a pseudo-corner cell. Since $D - Y$ is northwest, it cannot have any columns with a cell in the i_mth row and some row below the i_{m-1}th but no cell in the i_{m-1}th. Since the i_mth row of $D - Y$ does not contain a pseudo-corner cell, it follows that the i_mth row of $D - Y$ is contained in the i_{m-1}th. But this implies that the i_mth row of D is contained in the i_{m-1}th row of D. The argument at the end of Case 1 applies to finish the proof. ∎

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